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SUR LES REPRÉSENTATIONS UNITAIRES DES GROUPES DE LIE NILPOTENTS. III

J. DIXMIER

Dans un article antérieur **(3)**, j'ai donné quelques théorèmes généraux sur les représentations unitaires des groupes de Lie nilpotents simplement connexes. On va, dans le présent article, construire explicitement les représentations unitaires irréductibles et la formule de Plancherel des groupes de Lie nilpotents simplement connexes de dimension $\leqslant 5$.

Au § 1, on construit les algèbres de Lie nilpotentes (sur un corps quelconque) de dimension $\leqslant 5$. Ce travail a été fait indépendamment (mais non publié) par C. Chevalley. Différents auteurs se sont occupés depuis longtemps de cette classification (cf. par exemple **(7)**), mais je n'ai trouvé nulle part les tables de multiplication explicites.[1]

Pour toute algèbre de Lie $\mathfrak{g}$, on désignera par $\mathfrak{U}(\mathfrak{g})$ son algèbre enveloppante, par $\mathfrak{Z}(\mathfrak{g})$ le centre de $\mathfrak{U}(\mathfrak{g})$. Au § 2, on détermine $\mathfrak{Z}(\mathfrak{g})$ pour toutes les algèbres de Lie construites au § 1. On trouve, dans tous ces cas, que $\mathfrak{Z}(\mathfrak{g})$ est une algèbre de type fini (ce qu'on ignore en général); on constate même que $\mathfrak{Z}(\mathfrak{g})$ est une algèbre de polynômes, à l'exception d'un cas (il s'agit de l'algèbre notée plus loin $\mathfrak{g}_{5,5}$).

On peut alors déterminer l'ensemble Λ des caractères hermitiens de $\mathfrak{Z}(\mathfrak{g})$ pour les algèbres $\mathfrak{g}$ du § 1. Il s'identifie à une variété algébrique affine réelle Ω birationnellement équivalente à un espace affine (en fait, sauf dans le cas de $\mathfrak{g}_{5,5}$, Ω est même un espace affine). D'après la théorie générale de **(3)**, les représentations unitaires irréductibles du groupe simplement connexe correspondant sont paramétrées "en général" par les points de Ω. L'étude détaillée est faite du § 4 au § 11, et montre qu'il existe effectivement des caractères hermitiens ne correspondant à aucune représentation unitaire irréductible, des caractères hermitiens correspondant à un nombre fini > 1 de représentations unitaires irréductibles, et des caractères hermitiens correspondant à une infinité de représentations unitaires irréductibles. On constate, dans les cas étudiés, que les représentations "exceptionnelles" sont des représentations triviales sur certains sous-groupes de dimension $\geqslant 1$, et peuvent donc être considérées comme des représentations d'un groupe de dimension strictement plus petite; ceci permet de paramétrer *toutes* les représentations unitaires irréductibles, par récurrence sur la dimension du groupe. Malheureusement,

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[1]Je n'ai pu consulter la thèse de K. A. Umlauf (Ueber die Zusammensetzung der endlichen continuierlichen Transformationsgruppen, insbesondere der Gruppen vom Range Null, Leipzig, 1891), citée dans la thèse de E. Cartan.

ce fait n'est pas général, comme le montre l'exemple d'un groupe nilpotent de dimension 7 étudié au § 12. Toutefois, on peut espérer que la situation pour un groupe nilpotent simplement connexe Γ quelconque d'algèbre de Lie $\mathfrak{g}$ serait la suivante:

$\mathfrak{Z}(\mathfrak{g})$ est une algèbre de type fini; l'ensemble Λ des caractères hermitiens de $\mathfrak{Z}(\mathfrak{g})$ s'identifie à une variété algébrique affine Ω birationnellement équivalente à un espace affine; l'ensemble Ω' des points de Ω correspondant à une représentation unitaire irréductible et à une seule (à une équivalence près) de Γ est une partie non vide Ω' de Ω, ouverte pour la topologie de Zariski. Les points de $\Omega - \Omega'$ correspondent aux caractères de $\mathfrak{Z}(\mathfrak{g})$ nuls sur certains ideaux premiers $\mathfrak{a}$ de $\mathfrak{Z}(\mathfrak{g})$; soit $\mathfrak{a}'$ l'idéal (bilatère) de $\mathfrak{U}(\mathfrak{g})$ engendré par $\mathfrak{a}$; alors, les représentations unitaires irréductibles de Γ dont le caractère s'annule sur $\mathfrak{a}$ sont scalaires sur le centre de $\mathfrak{U}(\mathfrak{g})/\mathfrak{a}$; et ces représentations unitaires se paramètrent "en général" à l'aide des caractères de ce centre; il y a à nouveau des exceptions, mais, de proche en proche, on est ramené au bout d'un nombre fini de pas à une famille de représentations unitaires irréductibles sans sous-famille exceptionnelle.

L'exemple de $\mathfrak{g}_{6,6}$ montre d'ailleurs que Ω' n'est peut-être pas le sous-ensemble le plus intéressant de Ω: dans ce cas, en effet, la forme différentielle rationnelle qui intervient dans la formule de Plancherel n'est régulière que sur une partie de Ω'.

Au § 13, on montre que, pour un groupe de Lie nilpotent non simplement connexe, la formule de Plancherel doit faire intervenir les représentations unitaires irréductibles exceptionnelles.

On désignera par $\mathbf{R}$ le corps des nombres réels, par $\mathbf{C}$ le corps des nombres complexes, par $L^p_{\mathbf{C}}(\mathbf{R}^n)$ l'ensemble des fonctions complexes sur $\mathbf{R}^n$ de puissance p-ième intégrable pour la mesure de Lebesgue, par $\mathscr{S}(\mathbf{R}^n)$ l'ensemble des fonctions complexes sur $\mathbf{R}^n$ indéfiniment différentiables à décroissance rapide, par D_i l'opérateur de dérivation par rapport à la $i$ème variable dans $\mathscr{S}(\mathbf{R}^n)$, par M_i l'opérateur de multiplication par la $i$ème variable dans $\mathscr{S}(\mathbf{R}^n)$. Sur un groupe localement compact dont l'élément générique est noté γ, $d\gamma$ désignera une mesure de Haar quelconque.

1. Algèbres de Lie nilpotentes de dimension ≤ 5. Soit K un corps commutatif. Nous laissons au lecteur le soin de vérifier que les tables de multiplication ci-dessous définissent bien des algèbres de Lie nilpotentes sur K. (Dans ces tables, $(x_1, x_2, \ldots, x_n)$ désigne une base d'une algèbre de Lie de dimension n; on donne le crochet $[x_i, x_j]$ seulement pour $i < j$ et seulement si ce crochet est $\neq 0$.)

Dimension 3:

$$\mathfrak{g}_3 : [x_1, x_2] = x_3.$$

Dimension 4:

$$\mathfrak{g}_4 : [x_1, x_2] = x_3, \qquad [x_1, x_3] = x_4.$$

Dimension 5:

$\mathfrak{g}_{5,1} : [x_1, x_2] = x_5, \quad [x_3, x_4] = x_5.$
$\mathfrak{g}_{5,2} : [x_1, x_2] = x_4, \quad [x_1, x_3] = x_5.$
$\mathfrak{g}_{5,3} : [x_1, x_2] = x_4, \quad [x_1, x_4] = x_5, \quad [x_2, x_3] = x_5.$
$\mathfrak{g}_{5,4} : [x_1, x_2] = x_3, \quad [x_1, x_3] = x_4, \quad [x_2, x_3] = x_5.$
$\mathfrak{g}_{5,5} : [x_1, x_2] = x_3, \quad [x_1, x_3] = x_4, \quad [x_1, x_4] = x_5.$
$\mathfrak{g}_{5,6} : [x_1, x_2] = x_3, \quad [x_1, x_3] = x_4, \quad [x_1, x_4] = x_5, \quad [x_2, x_3] = x_5.$

Notons d'autre part $\mathfrak{g}_1$ l'unique algèbre de Lie de dimension 1 sur K. Alors:

PROPOSITION 1. *Toute algèbre de Lie nilpotente sur K de dimension ≤ 5 est isomorphe à l'une des algèbres du tableau suivant:*
Dimension 1: $\mathfrak{g}_1$.
Dimension 2: $(\mathfrak{g}_1)^2$.
Dimension 3: $(\mathfrak{g}_1)^3$, $\mathfrak{g}_3$.
Dimension 4: $(\mathfrak{g}_1)^4$, $\mathfrak{g}_3 \times \mathfrak{g}_1$, $\mathfrak{g}_4$.
Dimension 5: $(\mathfrak{g}_1)^5$, $\mathfrak{g}_3 \times (\mathfrak{g}_1)^2$, $\mathfrak{g}_4 \times \mathfrak{g}_1$, $\mathfrak{g}_{5,1}$, $\mathfrak{g}_{5,2}$, $\mathfrak{g}_{5,3}$, $\mathfrak{g}_{5,4}$, $\mathfrak{g}_{5,5}$, $\mathfrak{g}_{5,6}$.
Les algèbres de ce tableau sont deux à deux non isomorphes.

Démonstration. Montrons d'abord que ces algèbres sont deux à deux non isomorphes. Ceci est évident pour les dimensions 1, 2, 3. Les centres de $(\mathfrak{g}_1)^4$, $\mathfrak{g}_3 \times \mathfrak{g}_1$, $\mathfrak{g}_4$ sont respectivement de dimensions 4, 2, 1, donc ces algèbres sont deux à deux non isomorphes. Pour les algèbres de dimension 5, les dimensions des idéaux de la série centrale descendante et de la série centrale ascendante sont données par le tableau suivant:

$(\mathfrak{g}_1)^5$:	0	5
$\mathfrak{g}_3 \times (\mathfrak{g}_1)^2$:	1, 0	3, 5
$\mathfrak{g}_4 \times \mathfrak{g}_1$:	2, 1, 0	2, 3, 5
$\mathfrak{g}_{5,1}$:	1, 0	1, 5
$\mathfrak{g}_{5,2}$:	2, 0	2, 5
$\mathfrak{g}_{5,3}$:	2, 1, 0	1, 3, 5
$\mathfrak{g}_{5,4}$:	3, 2, 0	2, 3, 5
$\mathfrak{g}_{5,5}$:	3, 2, 1, 0	1, 2, 3, 5
$\mathfrak{g}_{5,6}$:	3, 2, 1, 0	1, 2, 3, 5.

On voit que ces algèbres sont deux à deux non isomorphes, à l'exception peut-être de $\mathfrak{g}_{5,5}$ et $\mathfrak{g}_{5,6}$. Mais l'annulateur de $[\mathfrak{g}_{5,5}, \mathfrak{g}_{5,5}]$ dans $\mathfrak{g}_{5,5}$ est de dimension 4, tandis que l'annulateur de $[\mathfrak{g}_{5,6}, \mathfrak{g}_{5,6}]$ dans $\mathfrak{g}_{5,6}$ est de dimension 3; donc $\mathfrak{g}_{5,5}$ et $\mathfrak{g}_{5,6}$ sont non isomorphes.

Montrons maintenant qu'on a bien obtenu toutes les algèbres de Lie nilpotentes de dimension ≤ 5. Posons, pour toute algèbre de Lie $\mathfrak{g}$, $D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, $D^2\mathfrak{g} = [D^1\mathfrak{g}, D^1\mathfrak{g}]$. Alors, si $\mathfrak{g}$ est nilpotente, on a dim $\mathfrak{g}/D^1\mathfrak{g} \geq 2$, dim $D^1\mathfrak{g}/D^2\mathfrak{g} \geq 3$ (cf. **2**). Ceci posé, la classification est évidente pour les dimensions 0, 1, 2. Soit $\mathfrak{g}$ une algèbre de Lie nilpotente de dimension 3 sur K. La dimension de $D^1\mathfrak{g}$ est 0 ou 1. Si dim $D^1\mathfrak{g} = 0$, $\mathfrak{g}$ est isomorphe à $(\mathfrak{g}_1)^3$; si

dim $D^1\mathfrak{g} = 1$, $D^1\mathfrak{g}$ est contenu dans le centre de $\mathfrak{g}$; prenant x_1 et x_2 dans $\mathfrak{g}$ linéairement indépendants modulo $D^1\mathfrak{g}$, $[x_1, x_2] = x_3$ engendre $D^1\mathfrak{g}$, et (x_1, x_2, x_3) est une base de $\mathfrak{g}$, donc $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_3$. Soit maintenant $\mathfrak{g}$ une algèbre de Lie nilpotente de dimension 4 sur K. La dimension de $D^1\mathfrak{g}$ est 0, 1 ou 2. Si dim $D^1\mathfrak{g} = 0$, $\mathfrak{g}$ est isomorphe à $(\mathfrak{g}_1)^4$. Supposons dim $D^1\mathfrak{g} = 1$. Alors, $\mathfrak{g}$ est extension centrale de l'algèbre de Lie abélienne $\mathfrak{g}/D^1\mathfrak{g}$, qui est de dimension 3, par $D^1\mathfrak{g}$; comme une forme bilinéaire alternée sur un espace de dimension 3 est dégénérée, on voit qu'il existe un sous-espace $\mathfrak{h}$ de dimension 1 de $\mathfrak{g}$ contenu dans le centre de $\mathfrak{g}$, avec $\mathfrak{h} \cap D^1\mathfrak{g} = \{0\}$; soit $\mathfrak{h}'$ un sous-espace vectoriel de $\mathfrak{g}$ de dimension 3 contenant $D^1\mathfrak{g}$ tel que $\mathfrak{g}$ soit somme directe de $\mathfrak{h}$ et $\mathfrak{h}'$; alors $\mathfrak{h}$ et $\mathfrak{h}'$ sont des idéaux, donc $\mathfrak{g}$ est isomorphe à $\mathfrak{h} \times \mathfrak{h}'$, c'est-à-dire, d'après ce qui précède, à $\mathfrak{g}_1 \times \mathfrak{g}_3$. Supposons maintenant dim $D^1\mathfrak{g} = 2$; alors $D^1\mathfrak{g}$ est une algèbre de Lie abélienne; il existe une base de $D^1\mathfrak{g}$ telle que, pour tout $x \in \mathfrak{g}$, $\mathrm{ad}_{D^1\mathfrak{g}}x$ admette une matrice de la forme

$$\begin{pmatrix} 0 & 0 \\ \lambda(x) & 0 \end{pmatrix};$$

et λ est une forme linéaire sur $\mathfrak{g}$ nulle sur $D^1\mathfrak{g}$; soit $\mathfrak{h}$ un sous-espace vectoriel de dimension 3 de $\mathfrak{g}$, contenant $D^1\mathfrak{g}$, contenu dans le noyau de λ; alors $\mathfrak{h}$ est un idéal abélien de $\mathfrak{g}$; soit x_1 un élément de $\mathfrak{g}$ n'appartenant pas à $\mathfrak{h}$; soit $u = \mathrm{ad}_{\mathfrak{h}}x_1$; on a $D^1\mathfrak{g} = u(\mathfrak{h})$, et u est nilpotent; utilisant pour u la forme réduite de Jordan, on voit que $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_4$.

Il nous reste à classer les algèbres de Lie nilpotentes de dimension 5. Si $\mathfrak{g}$ est une telle algèbre, $D^1\mathfrak{g}$ est de dimension 0, 1, 2, ou 3, et $D^1\mathfrak{g}$ est une algèbre de Lie abélienne. Si dim $D^1\mathfrak{g} = 0$, $\mathfrak{g}$ est isomorphe à $(\mathfrak{g}_1)^5$. Si dim $D^1\mathfrak{g} = 1$, $D^1\mathfrak{g}$ est dans le centre de $\mathfrak{g}$ et le crochet dans $\mathfrak{g}$ définit une forme bilinéaire alternée non nulle sur $\mathfrak{g}/D^1\mathfrak{g}$; suivant que cette forme est dégénérée ou non, $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_3 \times (\mathfrak{g}_1)^2$ ou à $\mathfrak{g}_{5,1}$. Supposons dim $D^1\mathfrak{g} = 2$, et soit $\mathfrak{c}$ le centre de $\mathfrak{g}$. Nous distinguerons trois cas:

(a) $D^1\mathfrak{g} \subset \mathfrak{c}$. Soit $\mathfrak{h} = \mathfrak{g}/D^1\mathfrak{g}$, qui est une algèbre de Lie abélienne de dimension 3. Le crochet dans $\mathfrak{g}$ définit une application bilinéaire alternée de $\mathfrak{h}$ dans $D^1\mathfrak{g}$, donc, après choix d'une base de $D^1\mathfrak{g}$, deux formes bilinéaires alternées f_1, f_2 sur $\mathfrak{h}$. Comme dim $\mathfrak{h} = 3$, il existe $x \in \mathfrak{h}$ tel que $f_1(x, y) = 0$ pour tout $y \in \mathfrak{h}$. On peut choisir y non proportionnel à x de façon que $f_2(x, y) = 0$. Soit $z \in \mathfrak{h}$, tel que x, y, z forment une base de $\mathfrak{h}$. Soient x', y', z' des représentants de x, y, z dans $\mathfrak{g}$. On a $[x', y'] = 0$, et $[x', z']$, $[y', z']$ engendrent $D^1\mathfrak{g}$. On voit alors que $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_{5,2}$.

(b) $D^1\mathfrak{g} \not\subset \mathfrak{c}$, $\mathfrak{c} \not\subset D^1\mathfrak{g}$. Soit $\mathfrak{h}$ un sous-espace de dimension 1 de $\mathfrak{c}$ non contenu dans $D^1\mathfrak{g}$. Soit $\mathfrak{h}'$ un sous-espace de dimension 4 de $\mathfrak{g}$ contenant $D^1\mathfrak{g}$ mais non $\mathfrak{h}$. Alors, $\mathfrak{h}$ et $\mathfrak{h}'$ sont deux idéaux de somme directe $\mathfrak{g}$, donc $\mathfrak{g}$ est isomorphe à $\mathfrak{h} \times \mathfrak{h}'$. On a $D^1\mathfrak{h}' = D^1\mathfrak{g}$, donc, d'après ce qu'on a vu plus haut, $\mathfrak{h}'$ est isomorphe à $\mathfrak{g}_4$. Donc $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_4 \times \mathfrak{g}_1$.

(c) $\mathfrak{c} \subset D^1\mathfrak{g}$, $\mathfrak{c} \neq D^1\mathfrak{g}$. Alors, $\mathfrak{c}$ est de dimension 1. L'algèbre de Lie $\mathfrak{g}/\mathfrak{c}$, de dimension 4, est telle que $D^1(\mathfrak{g}/\mathfrak{c}) = (D^1\mathfrak{g})/\mathfrak{c}$ soit de dimension 1, donc

est isomorphe à $\mathfrak{g}_3 \times \mathfrak{g}_1$ d'après ce qu'on a vu plus haut. Il existe une base (y_1, y_2, y_3, y_4) de $\mathfrak{g}/\mathfrak{c}$ telle que

$$[y_1, y_2] = y_3,\ [y_1, y_3] = [y_1, y_4] = [y_2, y_3] = [y_2, y_4] = [y_3, y_4] = 0.$$

Soient x_1, x_2, x_3, x_4 des représentants de y_1, y_2, y_3, y_4 dans $\mathfrak{g}$, et x_5 un élément non nul de $\mathfrak{c}$. En choisissant convenablement x_3, on a

$$[x_1, x_2] = x_3,\ [x_1, x_3] = \lambda x_5,\ [x_1, x_4] = \mu x_5,$$
$$[x_2, x_3] = \nu x_5,\ [x_2, x_4] = \rho x_5,\ [x_3, x_4] = \sigma x_5.$$

L'égalité

$$[[x_1, x_2], x_4] + [[x_2, x_4], x_1] + [[x_4, x_1], x_2] = 0$$

donne $\sigma = 0$. Comme $x_3 \notin \mathfrak{c}$, on a $\lambda \neq 0$ ou $\nu \neq 0$. Echangeant au besoin x_1 et x_2, on peut supposer $\lambda \neq 0$. Remplaçant x_2 par $x_2 - (\nu/\lambda)x_1$, on peut supposer $\nu = 0$. Remplaçant x_4 par $x_4 - (\mu/\lambda)x_2$, on peut supposer $\mu = 0$. Comme $x_4 \notin \mathfrak{c}$, on a $\rho \neq 0$. Multipliant x_4 et x_5 par des scalaires, on peut supposer $\lambda = \rho = 1$. On a alors la table de multiplication de $\mathfrak{g}_{5,3}$, à l'échange près de x_3 et x_4.

Enfin, $\mathfrak{g}$ étant toujours une algèbre de Lie nilpotente de dimension 5, supposons $\dim D^1\mathfrak{g} = 3$. Soit $\mathfrak{c}$ le centre de $\mathfrak{g}$. On a $\mathfrak{c} \subset D^1\mathfrak{g}$; sinon, d'après un raisonnement déjà fait, $\mathfrak{g}$ serait isomorphe au produit d'une algèbre de dimension 1 et d'une algèbre de dimension 4, et on aurait $\dim D^1\mathfrak{g} \leqslant 2$. On a $\mathfrak{c} \neq D^1\mathfrak{g}$, car, si on avait $[\mathfrak{g}, D^1\mathfrak{g}] = 0$, on aurait $\dim D^1\mathfrak{g} \leqslant 1$. Donc $\mathfrak{c}$ est de dimension 1 ou 2.

(a) Supposons $\dim \mathfrak{c} = 2$. Alors, $\mathfrak{g}/\mathfrak{c}$ est isomorphe à $\mathfrak{g}_3$. Il existe donc des éléments x_1, x_2, x_3 de $\mathfrak{g}$, linéairement indépendants modulo $\mathfrak{c}$, tels que

$$[x_1, x_2] = x_3, \qquad [x_1, x_3] \in \mathfrak{c}, \qquad [x_2, x_3] \in \mathfrak{c}.$$

Comme $\dim D^1\mathfrak{g} = 3$, $[x_1, x_3] = x_4$ et $[x_2, x_3] = x_5$ doivent former une base de $\mathfrak{c}$. On voit donc que $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_{5,4}$.

(b) Supposons $\dim \mathfrak{c} = 1$. Alors, $\mathfrak{g}/\mathfrak{c}$ est isomorphe à $\mathfrak{g}_4$, donc $\mathfrak{g}$ est extension centrale de $\mathfrak{g}_4$ par $\mathfrak{g}_1$. Cette extension est définie par un cocycle f, forme bilinéaire alternée sur $\mathfrak{g}_4$. Soit (e_1, e_2, e_3, e_4) une base de $\mathfrak{g}/\mathfrak{c} = \mathfrak{g}_4$ pour laquelle la table de multiplication est celle indiquée plus haut. Ecrivant que f est un cocycle, on obtient $f(e_2, e_4) = f(e_3, e_4) = 0$. En ajoutant à f un cobord, on peut supposer que $f(e_1, e_2) = f(e_1, e_3) = 0$. La table de multiplication de $\mathfrak{g}$ est alors (x_5 désignant un élément non nul de $\mathfrak{c}$ et x_1, x_2, x_3, x_4 des représentants de e_1, e_2, e_3, e_4 dans $\mathfrak{g}$):

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = x_4, \quad [x_1, x_4] = \lambda x_5,$$
$$[x_2, x_3] = \mu x_5, \quad [x_2, x_4] = 0, \quad [x_3, x_4] = 0.$$

Comme $x_4 \notin \mathfrak{c}$, on a $\lambda \neq 0$. Remplaçant x_5 par $\lambda^{-1}x_5$, on peut supposer $\lambda = 1$. Si $\mu = 0$, $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_{5,5}$. Supposons désormais $\mu \neq 0$. Remplaçant x_1, x_2, x_3, x_4, x_5 par $x_1, \mu^{-1}x_2, \mu^{-1}x_3, \mu^{-1}x_4, \mu^{-1}x_5$, on voit qu'on peut

supposer $\mu = 1$. Alors, $\mathfrak{g}$ est isomorphe à $\mathfrak{g}_{5,6}$. Ceci achève de prouver la proposition 1.

2. Centre de l'algèbre enveloppante pour les algèbres de Lie précédentes. Nous allons chercher $\mathfrak{Z}(\mathfrak{g})$ pour les algèbres de la proposition 1. Comme $\mathfrak{Z}(\mathfrak{g} \times \mathfrak{g}') = \mathfrak{Z}(\mathfrak{g}) \otimes \mathfrak{Z}(\mathfrak{g}')$, il suffit de le faire pour $\mathfrak{g}_3$, $\mathfrak{g}_4$, $\mathfrak{g}_{5,1}$, $\mathfrak{g}_{5,2}$, $\mathfrak{g}_{5,3}$, $\mathfrak{g}_{5,4}$, $\mathfrak{g}_{5,5}$, $\mathfrak{g}_{5,6}$.

PROPOSITION 2. *Supposons K de caractéristique* 0. *Utilisant les mêmes tables de multiplication qu'au* § 1, *on a le tableau suivant:*

$$
\begin{aligned}
\mathfrak{Z}(\mathfrak{g}_3) &= K[x_3],\\
\mathfrak{Z}(\mathfrak{g}_4) &= K[x_4, 2x_2 x_4 - x_3^2],\\
\mathfrak{Z}(\mathfrak{g}_{5,1}) &= K[x_5],\\
\mathfrak{Z}(\mathfrak{g}_{5,2}) &= K[x_4, x_5, x_2 x_5 - x_3 x_4],\\
\mathfrak{Z}(\mathfrak{g}_{5,3}) &= K[x_5],\\
\mathfrak{Z}(\mathfrak{g}_{5,4}) &= K[x_4, x_5, 2x_1 x_5 - 2x_2 x_4 + x_3^2],\\
\mathfrak{Z}(\mathfrak{g}_{5,5}) &= K[x_5, 2x_3 x_5 - x_4^2, 3x_2 x_5^2 - 3x_3 x_4 x_5 + x_4^3,\\
&\qquad 9x_2^2 x_5^2 - 18x_2 x_3 x_4 x_5 + 6x_2 x_4^3 + 8x_3^3 x_5 - 3x_3^2 x_4^2],\\
\mathfrak{Z}(\mathfrak{g}_{5,6}) &= K[x_5].
\end{aligned}
$$

A l'exception de $\mathfrak{Z}(\mathfrak{g}_{5,5})$, ces algèbres sont des algèbres de polynômes dont le tableau précédent donne des générateurs algébriquement indépendants. L'algèbre $\mathfrak{Z}(\mathfrak{g}_{5,5})$ n'est pas une algèbre de polynômes.

Démonstration. Pour toute algèbre de Lie $\mathfrak{g}$, nous désignerons par $\mathfrak{S}(\mathfrak{g})$ l'algèbre symétrique de l'espace vectoriel $\mathfrak{g}$, dans laquelle $\mathfrak{g}$ opère par des dérivations prolongeant les opérateurs de la représentation adjointe. Nous noterons $\mathfrak{J}(\mathfrak{g})$ la sous-algèbre de $\mathfrak{S}(\mathfrak{g})$ formée des éléments annulés par $\mathfrak{g}$. Les éléments de $\mathfrak{J}(\mathfrak{g}_3)$ sont les éléments f de $\mathfrak{S}(\mathfrak{g}_3)$ satisfaisant aux équations suivantes:

$$[x_1, x_2]f'_{x_2} + [x_1, x_3]f'_{x_3} = [x_2, x_1]f'_{x_1} + [x_2, x_3]f'_{x_3} = [x_3, x_1]f'_{x_1} + [x_3, x_2]f'_{x_2} = 0,$$

qui se réduisent à

$$f'_{x_1} = f'_{x_2} = 0.$$

Donc $\mathfrak{J}(\mathfrak{g}_3) = K[x_3]$. L'application canonique de $\mathfrak{J}(\mathfrak{g}_3)$ sur $\mathfrak{Z}(\mathfrak{g}_3)$ est ici un isomorphisme, et $\mathfrak{Z}(\mathfrak{g}_3) = K[x_3]$.

Pour $\mathfrak{J}(\mathfrak{g}_4)$, nous avons le système d'équations

$$x_3 f'_{x_2} + x_4 f'_{x_3} = -x_3 f'_{x_1} = -x_4 f'_{x_1} = 0.$$

Il est clair que $x_4 \in \mathfrak{J}(\mathfrak{g}_4)$ et que $2x_2x_4 - x_3^2 \in \mathfrak{J}(\mathfrak{g}_4)$. Soient $\mathfrak{h}_1 = Kx_4$, $\mathfrak{h}_2 = Kx_4 + Kx_3$, $\mathfrak{h}_3 = Kx_4 + Kx_3 + Kx_2$, qui forment une suite croissante d'idéaux de $\mathfrak{g}_4$. Les équations précédentes montrent facilement que

$$0 \neq \mathfrak{J}(\mathfrak{g}_4) \cap \mathfrak{S}(\mathfrak{h}_1) = \mathfrak{J}(\mathfrak{g}_4) \cap \mathfrak{S}(\mathfrak{h}_2) \neq \mathfrak{J}(\mathfrak{g}_4) \cap \mathfrak{S}(\mathfrak{h}_3) = \mathfrak{J}(\mathfrak{g}_4).$$

Appliquons le lemme 3 de **(3)**, où on considère $\mathfrak{g}$ comme l'algèbre de Lie abélienne sous-jacente à $\mathfrak{g}_4$, munie de l'algèbre de Lie des dérivations intérieures

de $\mathfrak{g}_4$. On voit que x_4 et $2x_2x_4 - x_3^2$ sont des éléments algébriquement indépendants engendrant le corps des fractions de $\mathfrak{I}(\mathfrak{g}_4)$, et que

$$\mathfrak{I}(\mathfrak{g}_4) \subset K[x_4, 2x_2 x_4 - x_3^2, x_4^{-1}].$$

Soit $f \in \mathfrak{I}(\mathfrak{g}_4)$. On a donc

$$f = x_4^{-s}[x_4^r g_r + x_4^{r-1} g_{r-1} + \ldots + g_0]$$

où $g_r, g_{r-1}, \ldots, g_0 \in K[2x_2 x_4 - x_3^2]$. Si $s > 0$, g_0 est divisible par x_4. Considérant dans g_0 les termes en x_3, on en conclut que $g_0 = 0$. Donc

$$f = x_4^{-s+1}[x_4^{r-1} g_r + x_4^{r-2} g_{r-1} + \ldots + g_1].$$

Recommençant le raisonnement de proche en proche, on arrive au cas où $s = 0$, ce qui prouve que

$$f \in K[x_4, 2x_2 x_4 - x_3^2].$$

Donc $\mathfrak{I}(\mathfrak{g}_4) = K[x_4, 2x_2 x_4 - x_3^2]$. Comme $\mathfrak{h}_3$ est abélien, l'application canonique de $\mathfrak{I}(\mathfrak{g}_4)$ sur $\mathfrak{Z}(\mathfrak{g}_4)$ est un isomorphisme, et $\mathfrak{Z}(\mathfrak{g}_4) = K[x_4, 2x_2 x_4 - x_3^2]$.

Pour $\mathfrak{g}_{5,1}$, $\mathfrak{g}_{5,3}$, $\mathfrak{g}_{5,6}$, nous avons les systèmes d'équations

$$x_5 f'_{x_2} = - x_5 f'_{x_1} = x_5 f'_{x_4} = - x_5 f'_{x_3} = 0,$$
$$x_4 f'_{x_2} + x_5 f'_{x_4} = - x_4 f'_{x_1} + x_5 f'_{x_3} = - x_5 f'_{x_2} = - x_5 f'_{x_1} = 0,$$
$$x_3 f'_{x_2} + x_4 f'_{x_3} + x_5 f'_{x_4} = - x_3 f'_{x_1} + x_5 f'_{x_3} = - x_4 f'_{x_1} - x_5 f'_{x_2} = - x_5 f'_{x_1} = 0,$$

qui donnent dans les trois cas $\mathfrak{I}(\mathfrak{g}) = K[x_5]$, $\mathfrak{Z}(\mathfrak{g}) = K[x_5]$.

Pour $\mathfrak{g}_{5,2}$, on a le système

$$x_4 f'_{x_2} + x_5 f'_{x_3} = - x_4 f'_{x_1} = - x_5 f'_{x_1} = 0.$$

Il est clair que $x_4 \in \mathfrak{I}(\mathfrak{g}_{5,2})$, $x_5 \in \mathfrak{I}(\mathfrak{g}_{5,2})$, $x_2 x_5 - x_3 x_4 \in \mathfrak{I}(\mathfrak{g}_{5,2})$. Soient $\mathfrak{h}_1 = Kx_5$, $\mathfrak{h}_2 = Kx_5 + Kx_4$, $\mathfrak{h}_3 = Kx_5 + Kx_4 + Kx_3$, $\mathfrak{h}_4 = Kx_5 + Kx_4 + Kx_3 + Kx_2$, qui forment une suite croissante d'idéaux de $\mathfrak{g}_{5,2}$. On a

$$0 \neq \mathfrak{I}(\mathfrak{g}_{5,2}) \cap \mathfrak{S}(\mathfrak{h}_1) \neq \mathfrak{I}(\mathfrak{g}_{5,2}) \cap \mathfrak{S}(\mathfrak{h}_2) = \mathfrak{I}(\mathfrak{g}_{5,2}) \cap \mathfrak{S}(\mathfrak{h}_3)$$
$$\neq \mathfrak{I}(\mathfrak{g}_{5,2}) \cap \mathfrak{S}(\mathfrak{h}_4) = \mathfrak{I}(\mathfrak{g}_{5,2}).$$

Raisonnant comme pour $\mathfrak{g}_4$, on voit que $x_5, x_4, x_2x_5 - x_3x_4$ sont des générateurs algébriquement indépendants du corps des fractions de $\mathfrak{I}(\mathfrak{g}_{5,2})$, et que

$$\mathfrak{I}(\mathfrak{g}_{5,2}) \subset K[x_4, x_5, x_2 x_5 - x_3 x_4, x_5^{-1}];$$

on voit ensuite qu'en fait $\mathfrak{I}(\mathfrak{g}_{5,2}) = K[x_4, x_5, x_2x_5 - x_3x_4]$. Comme $\mathfrak{h}_4$ est abélien, l'application canonique de $\mathfrak{I}(\mathfrak{g}_{5,2})$ sur $\mathfrak{Z}(\mathfrak{g}_{5,2})$ est un isomorphisme, et

$$\mathfrak{Z}(\mathfrak{g}_{5,2}) = K[x_4, x_5, x_2x_5 - x_3x_4].$$

Pour $\mathfrak{g}_{5,4}$, on a le système

$$x_3 f'_{x_2} + x_4 f'_{x_3} = - x_3 f'_{x_1} + x_5 f'_{x_3} = - x_4 f'_{x_1} - x_5 f'_{x_2} = 0.$$

Il est clair que $x_4 \in \mathfrak{I}(\mathfrak{g}_{5,4})$, $x_5 \in \mathfrak{I}(\mathfrak{g}_{5,4})$, $2x_1x_5 - 2x_2x_4 + x_3^2 \in \mathfrak{I}(\mathfrak{g}_{5,4})$. Soient

$$\mathfrak{h}_1 = Kx_5,\ \mathfrak{h}_2 = Kx_5 + Kx_4,\ \mathfrak{h}_3 = Kx_5 + Kx_4 + Kx_3,\ \mathfrak{h}_4 = Kx_5 + Kx_4 \\ Kx_3 + Kx_2,$$

qui forment une suite croissante d'idéaux de $\mathfrak{g}_{5,4}$. On a

$$0 \neq \mathfrak{I}(\mathfrak{g}_{5,4}) \cap \mathfrak{S}(\mathfrak{h}_1) \neq \mathfrak{I}(\mathfrak{g}_{5,4}) \cap \mathfrak{S}(\mathfrak{h}_2) = \mathfrak{I}(\mathfrak{g}_{5,4}) \cap \mathfrak{S}(\mathfrak{h}_3) \\ = \mathfrak{I}(\mathfrak{g}_{5,4}) \cap \mathfrak{S}(\mathfrak{h}_4) \neq \mathfrak{I}(\mathfrak{g}_{5,4}).$$

Les mêmes méthodes que précédemment montrent que $\mathfrak{I}(\mathfrak{g}_{5,4}) = K[x_4, x_5, 2x_1x_5 - 2x_2x_4 + x_3^2]$. Les images canoniques dans $\mathfrak{U}(\mathfrak{g}_{5,4})$ de $x_4, x_5, 2x_1x_5 - 2x_2x_4 + x_3^2$ (considérés comme éléments de $\mathfrak{S}(\mathfrak{g}_{5,4})$) sont $x_4, x_5, 2x_1x_5 - 2x_2x_4 + x_3^2$ (considérés comme éléments de $\mathfrak{U}(\mathfrak{g}_{5,4})$). J'ignore si l'application canonique de $\mathfrak{I}(\mathfrak{g}_{5,4})$ sur $\mathfrak{Z}(\mathfrak{g}_{5,4})$ est un isomorphisme. Mais nous allons voir cependant que x_4, x_5, $2x_1x_5 - 2x_2x_4 + x_3^2$ engendrent $\mathfrak{Z}(\mathfrak{g}_{5,4})$, par un raisonnement applicable à toute algèbre de Lie et certainement connu (il m'a été, en tous cas, signalé il y a longtemps par H. Cartan). Soit $a \in \mathfrak{Z}(\mathfrak{g}_{5,4})$, et montrons que

$$a \in K[x_4, x_5, 2x_1 x_5 - 2x_2 x_4 + x_3^2].$$

Nous supposerons ce point établi pour les éléments de $\mathfrak{Z}(\mathfrak{g}_{5,4})$ dont la filtration (dans $\mathfrak{U}(\mathfrak{g}_{5,4})$) est strictement inférieure à la filtration n de a. Soit f l'image canonique de a dans $\mathfrak{I}(\mathfrak{g}_{5,4})$. On a

$$f = \Sigma_j \lambda_j x_4^{m_j} x_5^{n_j} (2x_1 x_5 - 2x_2 x_4 + x_3^2)^{p_j}$$

avec $m_j + n_j + p_j < n$ pour tout j. Donc a est congru, modulo les éléments de $\mathfrak{U}(\mathfrak{g}_{5,4})$ de filtration $< n$, à

$$\Sigma_j \lambda_j x_4^{m_j} x_5^{n_j} (2x_1 x_5 - 2x_2 x_4 + x_3^2)^{p_j},$$

(calculé dans $\mathfrak{U}(\mathfrak{g}_{5,4})$). Alors,

$$a - \Sigma_j \lambda_j x_4^{m_j} x_5^{n_j} (2x_1 x_5 - 2x_2 x_4 + x_3^2)^{p_j}$$

est un élément de $\mathfrak{Z}(\mathfrak{g}_{5,4})$ qui appartient à $K[x_4, x_5, 2x_1x_5 - 2x_2x_4 + x_3^2]$ d'après l'hypothèse de récurrence. D'où notre assertion. Ainsi,

$$\mathfrak{Z}(\mathfrak{g}_{5,4}) = K[x_4, x_5, 2x_1 x_5 - 2x_2 x_4 + x_3^2],$$

et $x_4, x_5, 2x_1x_5 - 2x_2x_4 + x_3^2$ sont algébriquement indépendants d'après le lemme 3 de **(3)**.

Pour $\mathfrak{g}_{5,5}$, on a le système

$$x_3 f'_{x_2} + x_4 f'_{x_3} + x_5 f'_{x_4} = -x_3 f'_{x_1} = -x_4 f'_{x_1} = -x_5 f'_{x_1} = 0.$$

Ce cas est essentiellement traité dans **(8**, p. 244**)**, mais nous allons procéder directement. On vérifie que

$$x_5 \in \mathfrak{I}(\mathfrak{g}_{5,5}),$$
$$f_1 = 2x_3x_5 - x_4^2 \in \mathfrak{I}(\mathfrak{g}_{5,5}),$$
$$f_2 = 3x_2x_5^2 - 3x_3x_4x_5 + x_4^3 \in \mathfrak{I}(\mathfrak{g}_{5,5}).$$

Soient $\mathfrak{h}_1 = Kx_5$, $\mathfrak{h}_2 = Kx_5 + Kx_4$, $\mathfrak{h}_3 = Kx_5 + Kx_4 + Kx_3$, $\mathfrak{h}_4 = Kx_5 + Kx_4 + Kx_3 + Kx_2$, qui forment une suite croissante d'idéaux de $\mathfrak{g}_{5,5}$. On a

$$0 \neq \mathfrak{J}(\mathfrak{g}_{5,5}) \cap \mathfrak{S}(\mathfrak{h}_1) = \mathfrak{J}(\mathfrak{g}_{5,5}) \cap \mathfrak{S}(\mathfrak{h}_2) \neq \mathfrak{J}(\mathfrak{g}_{5,5}) \cap \mathfrak{S}(\mathfrak{h}_3) \\ \neq \mathfrak{J}(\mathfrak{g}_{5,5}) \cap \mathfrak{S}(\mathfrak{h}_4) = \mathfrak{J}(\mathfrak{g}_{5,5}).$$

Donc x_5, f_1, f_2 sont des générateurs algébriquement indépendants du corps des fractions de $\mathfrak{J}(\mathfrak{g}_{5,5})$, et $\mathfrak{J}(\mathfrak{g}_{5,5}) \subset K[x_5, f_1, f_2, x_5^{-1}]$. Soit

$$f_3 = x_5^{-2}(f_1^3 + f_2^2) = 9x_3^2x_5^2 - 18x_2x_3x_4x_5 + 6x_3x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2 \in \mathfrak{J}(\mathfrak{g}_{5,5}).$$

Nous allons montrer que $\mathfrak{J}(\mathfrak{g}_{5,5}) = K[x_5, f_1, f_2, f_3]$. Nous aurons besoin pour cela du lemme suivant:

LEMME 1. *Soit P un polynôme à 3 variables X, Y, Z, à coefficients dans K. Si $P(f_1, f_2, f_3)$ est divisible par x_5, $P(X, Y, Z)$ est divisible par $X^3 + Y^2$.*

En effet, soit $\mathfrak{a}$ l'idéal de $\mathfrak{S}(\mathfrak{g}_{5,5})$ engendré par x_5. Identifions canoniquement $\mathfrak{S}(\mathfrak{g}_{5,5})/\mathfrak{a}$ à $\mathfrak{S}(Kx_1 + Kx_2 + Kx_3 + Kx_4)$. Les images canoniques de f_1, f_2, f_3 dans $\mathfrak{S}(\mathfrak{g}_{5,5})/\mathfrak{a}$ sont

$$-x_4^2,\ x_4^3,\ 3x_4^2(2x_2x_4 - x_3^2).$$

D'après l'hypothèse du lemme, $P(-x_4^2, x_4^3, 3x_4^2(2x_2x_4 - x_3^2))$ est identiquement nul. Soit

$$P(X, Y, Z) = Z^rP_r(X, Y) + Z^{r-1}P_{r-1}(X, Y) + \ldots + P_0(X, Y).$$

Considérant les termes en x_2, on voit que

$$P_r(-x_4^2, x_4^3), P_{r-1}(-x_4^2, x_4^3), \ldots, P_0(-x_4^2, x_4^3)$$

sont identiquement nuls. Donc $P_r(X, Y), \ldots, P_0(X, Y)$ s'annulent sur la courbe d'équation $X^3 + Y^2 = 0$ (dans une clôture algébrique de K), et par suite sont divisibles par $X^3 + Y^2$. Donc $P(X, Y, Z)$ est divisible par $X^3 + Y^2$.

Revenons à la situation qui précéde le lemme 1. Nous allons montrer que si un élément de $K[x_5, f_1, f_2, f_3]$ est divisible par x_5^s (s entier $\geqslant 0$), son quotient f par x_5^s appartient à $K[x_5, f_1, f_2, f_3]$. Ecrivons en effet

$$f = x_5^{-s}[x_5^rg_r + x_5^{r-1}g_{r-1} + \ldots + g_0]$$

où $g_r, g_{r-1}, \ldots, g_0$ appartiennent à $K[f_1, f_2, f_3]$. Si $s > 0$, g_0 est divisible par x_5. D'après le lemme 1, $g_0 = (f_1^3 + f_2^2)g_0' = x_5^2f_3g_0'$, avec $g_0' \in K[f_1, f_2, f_3]$. Donc

$$f = x_5^{-s+1}[x_5^{r-1}g_r + x_5^{r-2}g_{r-1} + \ldots + x_5(g_2 + f_3g_0') + g_1].$$

Recommençant le raisonnement de proche en proche, on arrive au cas où $s = 0$, ce qui prouve que

$$f \in K[x_5, f_1, f_2, f_3].$$

En particulier, tout élément de $K[x_5, f_1, f_2, x_5^{-1}]$ qui est un polynôme, donc tout élément de $\mathfrak{J}(\mathfrak{g}_{5,5})$, appartient à $K[x_5, f_1, f_2, f_3]$. Donc $\mathfrak{J}(\mathfrak{g}_{5,5}) = K[x_5, f_1, f_2, f_3]$. Comme $\mathfrak{h}_4$ est abélien, l'application canonique de $\mathfrak{J}(\mathfrak{g}_{5,5})$ sur $\mathfrak{Z}(\mathfrak{g}_{5,5})$ est un isomorphisme, et on a

$$\mathfrak{Z}(\mathfrak{g}_{5,5}) = K[x_5, 2x_3x_5 - x_4^2, 3x_2x_5^2 - 3x_3x_4x_5 + x_4^3, \\ 9x_3^2x_5^2 - 18x_2x_3x_4x_5 + 6x_3x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2].$$

Enfin, $K[x_5, f_1, f_2]$ et son corps des fractions sont respectivement isomorphes à $K[X, Y, Z]$ et $K(X, Y, Z)$ (X, Y, Z désignant des indéterminées). Comme $f_3 = x_5^{-2}(f_1^3 + f_2^2)$, on voit que $\mathfrak{J}(\mathfrak{g}_{5,5})$ est isomorphe à $K[X, Y, Z, X^{-2}(Y^3 + Z^2)]$, c'est-à-dire à l'anneau des fonctions régulières sur la variété d'équation $TX^2 = Y^3 + Z^2$ dans l'espace affine K^4. Or cette variété n'est pas birégulièrement isomorphe à un espace affine puisqu'elle possède un point singulier à l'origine. Donc $\mathfrak{J}(\mathfrak{g}_{5,5})$, et par suite $\mathfrak{Z}(\mathfrak{g}_{5,5})$, ne sont pas isomorphes à des algèbres de polynômes.

3. Groupes de Lie nilpotents simplement connexes de dimension ≤ 5. Supposons désormais $K = \mathbf{R}$, et cherchons les groupes de Lie (réels) nilpotents simplement connexes de dimension ≤ 5. Si $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}''$, et si Γ, Γ', Γ'' sont les groupes de Lie simplement connexes correspondant à $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}''$, on a $\Gamma = \Gamma' \times \Gamma''$. Il nous suffit donc de chercher les groupes $\Gamma_1, \Gamma_3, \Gamma_4, \Gamma_{5,1}, \ldots, \Gamma_{5,6}$ correspondant à $\mathfrak{g}_1, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_{5,1}, \ldots, \mathfrak{g}_{5,6}$. Il est clair que Γ_1 est isomorphe à $\mathbf{R}$.

Pour construire Γ_3, nous cherchons d'abord des formes différentielles linéairement indépendantes $\omega_1, \omega_2, \omega_3$ de degré 1, à 3 variables ρ_1, ρ_2, ρ_3, vérifiant les équations de Maurer-Cartan qui sont ici

$$d\omega_1 = 0, \qquad d\omega_2 = 0, \qquad d\omega_3 = \omega_1 \wedge \omega_2.$$

On peut prendre $\omega_1 = d\rho_1$, $\omega_2 = d\rho_2$, $\omega_3 = \rho_1 d\rho_2 + d\rho_3$. La loi de groupe s'obtient alors en intégrant les équations $d\rho_1' = d\rho_1$, $d\rho_2' = d\rho_2$, $\rho_1' d\rho_2' + d\rho_3' = \rho_1 d\rho_2 + d\rho_3$, d'où facilement

$$\rho_1' = \rho_1 + \sigma_1, \qquad \rho_2' = \rho_2 + \sigma_2, \qquad \rho_3' = \rho_3 + \sigma_3 - \rho_2\sigma_1.$$

Donc

(1) $$(\sigma_1, \sigma_2, \sigma_3) \cdot (\rho_1, \rho_2, \rho_3) = (\rho_1 + \sigma_1,\ \rho_2 + \sigma_2,\ \rho_3 + \sigma_3 - \rho_2\sigma_1).$$

Les calculs étant purement mécaniques, nous nous contentons de les résumer pour les autres groupes.

Cas de Γ_4.

$$d\omega_1 = 0,\quad d\omega_2 = 0,\quad d\omega_3 = \omega_1 \wedge \omega_2,\quad d\omega_4 = \omega_1 \wedge \omega_3,$$

$$\omega_1 = d\rho_1,\quad \omega_2 = d\rho_2,\quad \omega_3 = \rho_1 d\rho_2 + d\rho_3,\quad \omega_4 = \tfrac{1}{2}\rho_1^2 d\rho_2 + \rho_1 d\rho_3 + d\rho_4,$$

(2) $$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \cdot (\rho_1, \rho_2, \rho_3, \rho_4) = (\rho_1 + \sigma_1,\ \rho_2 + \sigma_2,\ \rho_3 + \sigma_3 - \rho_2\sigma_1,\ \rho_4 + \sigma_4 - \rho_3\sigma_1 + \tfrac{1}{2}\rho_2\sigma_1^2).$$

Cas de $\Gamma_{5,1}$.

$$d\omega_1 = 0,\quad d\omega_2 = 0,\quad d\omega_3 = 0,\quad d\omega_4 = 0,$$

$$d\omega_5 = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4,$$

$$\omega_1 = d\rho_1,\quad \omega_2 = d\rho_2,\quad \omega_3 = d\rho_3,\quad \omega_4 = d\rho_4,\quad \omega_5 = \rho_1 d\rho_2 + \rho_3 d\rho_4 + d\rho_5$$

(3) $$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \cdot (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) = (\rho_1 + \sigma_1,\ \rho_2 + \sigma_2,\ \rho_3 + \sigma_3,\ \rho_4 + \sigma_4,\ \rho_5 + \sigma_5 - \rho_2\sigma_1 - \rho_4\sigma_3).$$

Cas de $\Gamma_{5,2}$.

$$d\omega_1 = 0,\quad d\omega_2 = 0,\quad d\omega_3 = 0,\quad d\omega_4 = \omega_1 \wedge \omega_2,\quad d\omega_5 = \omega_1 \wedge \omega_3,$$

$$\omega_1 = d\rho_1,\quad \omega_2 = d\rho_2,\quad \omega_3 = d\rho_3,\quad \omega_4 = \rho_1 d\rho_2 + d\rho_4,\quad \omega_5 = \rho_1 d\rho_3 + d\rho_5,$$

(4) $$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \cdot (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) = (\rho_1 + \sigma_1,\ \rho_2 + \sigma_2,\ \rho_3 + \sigma_3,\ \rho_4 + \sigma_4 - \rho_2\sigma_1,\ \rho_5 + \sigma_5 - \rho_3\sigma_1).$$

Cas de $\Gamma_{5,3}$.

$$d\omega_1 = 0, \quad d\omega_2 = 0, \quad d\omega_3 = 0, \quad d\omega_4 = \omega_1 \wedge \omega_2, \quad d\omega_5 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3,$$
$$\omega_1 = d\rho_1, \quad \omega_2 = d\rho_2, \quad \omega_3 = d\rho_3, \quad \omega_4 = \rho_1 d\rho_2 + d\rho_4,$$
$$\omega_5 = \rho_2 d\rho_3 + \rho_1 d\rho_4 + \tfrac{1}{2}\rho_1^2 d\rho_2 + d\rho_5,$$
$$\begin{aligned}(5)\qquad &(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \cdot (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \\ &= (\rho_1 + \sigma_1, \rho_2 + \sigma_2, \rho_3 + \sigma_3, \rho_4 + \sigma_4 - \rho_2\sigma_1, \rho_5 + \sigma_5 - \rho_3\sigma_2 - \tfrac{1}{2}\rho_4\sigma_1 \\ &\qquad + \tfrac{1}{2}\rho_2\sigma_1^2).\end{aligned}$$

Cas de $\Gamma_{5,4}$.

$$d\omega_1 = 0, \quad d\omega_2 = 0, \quad d\omega_3 = \omega_1 \wedge \omega_2, \quad d\omega_4 = \omega_1 \wedge \omega_3, \quad d\omega_5 = \omega_2 \wedge \omega_3,$$
$$\omega_1 = d\rho_1, \quad \omega_2 = d\rho_2, \quad \omega_3 = \rho_1 d\rho_2 + d\rho_3, \quad \omega_4 = \tfrac{1}{2}\rho_1^2 d\rho_2 + \rho_1 d\rho_3 + d\rho_4,$$
$$\omega_5 = \rho_2 d\rho_3 + d\rho_5,$$
$$\begin{aligned}(6)\qquad &(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \cdot (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \\ &= (\rho_1 + \sigma_1, \rho_2 + \sigma_2, \rho_3 + \sigma_3 - \rho_2\sigma_1, \rho_4 + \sigma_4 - \rho_3\sigma_1 + \tfrac{1}{2}\rho_2\sigma_1^2, \rho_5 + \sigma_5 \\ &\qquad + \tfrac{1}{2}\rho_2^2\sigma_1 - \rho_3\sigma_2 + \rho_2\sigma_1\sigma_2).\end{aligned}$$

Cas de $\Gamma_{5,5}$.

$$d\omega_1 = 0, \quad d\omega_2 = 0, \quad d\omega_3 = \omega_1 \wedge \omega_2, \quad d\omega_4 = \omega_1 \wedge \omega_3, \quad d\omega_5 = \omega_1 \wedge \omega_4,$$
$$\omega_1 = d\rho_1, \quad \omega_2 = d\rho_2, \quad \omega_3 = \rho_1 d\rho_2 + d\rho_3, \quad \omega_4 = \tfrac{1}{2}\rho_1^2 d\rho_2 + \rho_1 d\rho_3 + d\rho_4,$$
$$\omega_5 = \tfrac{1}{6}\rho_1^3 d\rho_2 + \tfrac{1}{2}\rho_1^2 d\rho_3 + \rho_1 d\rho_4 + d\rho_5,$$
$$\begin{aligned}(7)\qquad &(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \cdot (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \\ &= (\rho_1 + \sigma_1, \rho_2 + \sigma_2, \rho_3 + \sigma_3 - \rho_2\sigma_1, \rho_4 + \sigma_4 + \tfrac{1}{2}\rho_2\sigma_1^2 - \rho_3\sigma_1, \\ &\qquad \rho_5 + \sigma_5 - \tfrac{1}{6}\rho_2\sigma_1^3 + \tfrac{1}{2}\rho_3\sigma_1^2 - \rho_4\sigma_1).\end{aligned}$$

Cas de $\Gamma_{5,6}$.

$$d\omega_1 = 0, \; d\omega_2 = 0, \; d\omega_3 = \omega_1 \wedge \omega_2, \; d\omega_4 = \omega_1 \wedge \omega_3, \; d\omega_5 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3.$$
$$\omega_1 = d\rho_1, \quad \omega_2 = d\rho_2, \quad \omega_3 = \rho_1 d\rho_2 + d\rho_3, \quad \omega_4 = \tfrac{1}{2}\rho_1^2 d\rho_2 + \rho_1 d\rho_3 + d\rho_4,$$
$$\omega_5 = \tfrac{1}{6}\rho_1^3 d\rho_2 + \tfrac{1}{2}\rho_1^2 d\rho_3 + \rho_1 d\rho_4 + \rho_2 d\rho_3 + d\rho_5,$$
$$\begin{aligned}(8)\qquad &(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \cdot (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \\ &= (\rho_1 + \sigma_1, \rho_2 + \sigma_2, \rho_3 + \sigma_3 - \rho_2\sigma_1, \rho_4 + \sigma_4 + \tfrac{1}{2}\rho_2\sigma_1^2 - \rho_3\sigma_1, \\ &\qquad \rho_5 + \sigma_5 - \tfrac{1}{6}\rho_2\sigma_1^3 + \tfrac{1}{2}\rho_3\sigma_1^2 - \rho_4\sigma_1 + \tfrac{1}{2}\rho_2^2\sigma_1 - \rho_3\sigma_2 + \rho_2\sigma_1\sigma_2).\end{aligned}$$

Dans ce qui suit, nous allons déterminer les représentations unitaires irréductibles des groupes Γ_3, Γ_4, $\Gamma_{5,1}, \ldots, \Gamma_{5,6}$, et la formule de Plancherel pour ces groupes. La classification des représentations unitaires irréductibles pourrait se faire en utilisant la théorie de Mackey **(5)**, mais nous utiliserons systématiquement **(3)**. Notons aussi que la classification des représentations unitaires irréductibles de Γ_3 et $\Gamma_{5,1}$ a été faite par von Neumann **(6)**, et que la formule de Plancherel pour Γ_3 est due à Godement **(4)**.

4. Représentations unitaires irréductibles de Γ_3. Nous utilisons la table de multiplication de $\mathfrak{g}_3$ donnée au paragraphe 1. Soient $\mathfrak{h} = \mathbf{R}x_3$, Δ le sous-groupe correspondant de Γ_3, qui est à la fois le centre et le groupe des commutateurs de Γ_3.

D'après la proposition 2, et le lemme 15 de **(3)**, il y a correspondance biunivoque entre les caractères hermitiens χ de $\mathfrak{Z}(\mathfrak{g}_3)$ et les nombres réels λ. Cette correspondance est définie par la formule $\chi(x_3) = i\lambda$.

PROPOSITION 3. (i) *Pour tout nombre réel $\lambda \neq 0$, il existe une représentation unitaire irréductible U_λ de Γ_3 et (à une équivalence près) une seule dont le caractère prend la valeur $i\lambda$ en x_3. La représentation U_λ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$, l'ensemble des vecteurs indéfiniment différentiables est $\mathscr{S}(\mathbf{R})$. On a*

$$(9) \qquad U_\lambda(x_1) = D_1 \qquad U_\lambda(x_2) = i\lambda M_1 \qquad U_\lambda(x_3) = i\lambda.$$

Si γ est l'élément de Γ_3 de coordonnées (ρ_1, ρ_2, ρ_3) et si $f \in L_{\mathbf{C}}^2(\mathbf{R})$, on a

$$(10) \qquad (U_\lambda(\gamma)f)(\theta) = [\exp i\lambda(\rho_3 - \rho_2\theta)]\, f(\theta + \rho_1).$$

Si $F \in L_{\mathbf{C}}^1(\Gamma_3) \cap L_{\mathbf{C}}^2(\Gamma_3)$, on a

$$(11) \qquad \int_{\Gamma_3} |F(\gamma)|^2 d\gamma = \int_{\lambda\neq 0} tr(U_\lambda(F)^* U_\lambda(F))\,|\lambda|\, d\lambda.$$

(ii) *Il existe une infinité de représentations unitaires irréductibles de Γ_3, deux à deux non équivalentes, dont le caractère prend la valeur 0 en x_3. Ce sont les représentations triviales sur Δ; elles s'identifient donc aux représentations unitaires de dimension 1 du groupe abélien Γ_3/Δ.*

Démonstration. Soient $\mathfrak{h}' = \mathbf{R}x_3 + \mathbf{R}x_2$, qui est un idéal abélien de $\mathfrak{g}_3$, et Δ' le sous-groupe correspondant de Γ_3. Appliquons le lemme 24 de **(3)** à Γ_3 et Δ'; on peut prendre dans ce lemme $a_1 = x_3$, $a_2 = x_2$, $a = 1$, $b = x_3$. On voit que x_3 est classifiant pour $\mathfrak{g}_3$. Si $\lambda \neq 0$, il existe une représentation unitaire irréductible U_λ de Γ_3 et (à une équivalence près) une seule dont le caractère prend la valeur $i\lambda$ en x_3; compte tenu des lemmes 23 et 24 de **(3)**, U_λ est induite par n'importe quelle représentation unitaire de dimension 1 de Δ' dont le caractère prend la valeur $i\lambda$ en x_3, par exemple par la représentation dont le caractère prend les valeurs $i\lambda$ en x_3 et 0 en x_2. D'après les lemmes 29 et 31 de **(3)**, U_λ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$, l'ensemble des vecteurs indéfiniment différentiables pour U_λ est $\mathscr{S}(\mathbf{R})$, et on a les formules (9). La formule (10) résulte de la définition des représentations induites et du calcul suivant:

$$(\theta, 0, 0)\,(\rho_1, \rho_2, \rho_3) = (\theta + \rho_1, \rho_2, \rho_3 - \theta\rho_2) = (0, \rho_2, \rho_3 - \theta\rho_2)\,(\theta + \rho_1, 0, 0).$$

Maintenant, utilisons, dans **(3)**, la partie 2° de la démonstration du théorème 4. On peut y prendre $q = 1$, $a_1 = x_3$, $a_2 = x_2$, $b = x_3$, $F' \equiv 1$, $R_1(\lambda_1) = \lambda_1$. D'où la formule (11). Enfin, la partie (ii) de l'énoncé est évidente.

5. Représentations unitaires irréductibles de Γ_4. Nous utilisons la table de multiplication de $\mathfrak{g}_4$ donnée au § 1. Soient $\mathfrak{h} = \mathbf{R}x_4$, $\mathfrak{h}' = \mathbf{R}x_4 + \mathbf{R}x_3$, Δ et Δ' les sous-groupes correspondants de Γ_4. Alors Δ est le centre de Γ_4 et Δ' est le groupe des commutateurs de Γ_4.

D'après la proposition 2, et le lemme 15 de **(3)**, il y a correspondance biunivoque entre les caractères hermitiens χ de $\mathfrak{Z}(\mathfrak{g}_4)$ et les couples (λ, μ) de nombres réels. Cette correspondance est définie par les formules $\chi(x_4) = i\lambda$, $\chi(2x_2 x_4 - x_3^2) = \mu$.

PROPOSITION 4. *Soient λ et μ des nombres réels.*

(i) *Si $\lambda \neq 0$, il existe une représentation unitaire irréductible $U_{\lambda,\mu}$ de Γ_4 et (à une équivalence près) une seule dont le caractère prend la valeur $i\lambda$ en x_4 et la valeur μ en $2x_2x_4 - x_3^2$. La représentation $U_{\lambda,\mu}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$. L'ensemble des vecteurs indéfiniment différentiables est $\mathscr{S}(\mathbf{R})$. On a*

$$(12)\quad U_{\lambda,\mu}(x_1) = D_1,\quad U_{\lambda,\mu}(x_2) = -\tfrac{1}{2}i\frac{\mu}{\lambda} + \tfrac{1}{2}i\lambda M_1^2,\quad U_{\lambda,\mu}(x_3) = i\lambda M_1,$$
$$U_{\lambda,\mu}(x_4) = i\lambda.$$

Si γ est l'élément de Γ_4 de coordonnées $(\rho_1, \rho_2, \rho_3, \rho_4)$, et si $f \in L_{\mathbf{C}}^2(\mathbf{R})$, on a

$$(13)\quad (U_{\lambda,\mu}(\gamma)f)(\theta) = [\exp i(-\tfrac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2)]f(\theta + \rho_1).$$

Si $F \in L_{\mathbf{C}}^1(\Gamma_4) \cap L_{\mathbf{C}}^2(\Gamma_4)$, on a

$$(14)\quad \int_{\Gamma_4} |F(\gamma)|^2 d\gamma = \iint_{\lambda\neq 0} tr(U_{\lambda,\mu}(F)^* U_{\lambda,\mu}(F))d\lambda d\mu.$$

(ii) *Si $\lambda = 0$ et $\mu < 0$, il n'existe aucune représentation unitaire irréductible de Γ_4 dont le caractère prend la valeur $i\lambda$ en x_4 et μ en $2x_2x_4 - x_3^2$.*

(iii) *Si $\lambda > 0$ et $\mu > 0$, il existe (à une équivalence près) deux représentations unitaires irréductibles de Γ_4 dont le caractère prend la valeur $i\lambda$ en x_4 et μ en $2x_2x_4 - x_3^2$. Ces représentations sont triviales sur Δ, donc (comme Γ_4/Δ est isomorphe à Γ_3) s'identifient à des représentations de Γ_3, à savoir les représentations notées $U_{\pm\sqrt{\mu}}$ dans la proposition 3.*

(iv) *Si $\lambda = \mu = 0$, il existe une infinité de représentations unitaires irréductibles de Γ_4, deux à deux non équivalentes, dont le caractère prend la valeur $i\lambda$ en x_4 et μ en $2x_2x_4 - x_3^2$. Ce sont les représentations triviales sur Δ'; elles s'identifient donc aux représentations unitaires de dimension 1 du groupe abélien Γ_4/Δ'.*

Démonstration. Soient $\mathfrak{h}'' = \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}x_2$, qui est un idéal abélien de $\mathfrak{g}_4$, et Δ'' le sous-groupe correspondant de Γ_4. Appliquons le lemme 24 de **(3)** à Γ_4 et Δ''; on peut prendre dans ce lemme $a_1 = x_4$, $a_2 = 2x_2x_4 - x_3^2$, $a_3 = x_3$, $a = 1$, $b = x_4$. On voit que x_4 est classifiant pour $\mathfrak{g}_4$. Si $\lambda \neq 0$, il existe une représentation unitaire irréductible $U_{\lambda,\mu}$ de Γ_4 et (à une équivalence près) une seule dont le caractère prend les valeurs $i\lambda$ en x_4 et μ en $2x_2x_4 - x_3^2$; compte tenu des lemmes 23 et 24 de **(3)**, $U_{\lambda,\mu}$ est induite par n'importe quelle représentation unitaire de dimension 1 de Δ'' dont le caractère prend les valeurs $i\lambda$ en x_4 et μ en $2x_2x_4 - x_3^2$, par exemple par la représentation dont le caractère prend les valeurs $i\lambda$ en x_4, 0 en x_3 et $-\frac{1}{2}i\mu/\lambda$ en x_2. D'après les lemmes 29 et 31 de **(3)**, $U_{\lambda,\mu}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$, l'ensemble des vecteurs indéfiniment différentiables pour $U_{\lambda,\mu}$ est $\mathscr{S}(\mathbf{R})$, et on a les formules (12). La formule (13) résulte da la définition des représentations induites et du calcul suivant:

$$\begin{aligned}(\theta, 0, 0, 0)\ (\rho_1, \rho_2, \rho_3, \rho_4) &= (\theta + \rho_1, \rho_2, \rho_3 - \theta\rho_2, \rho_4 - \theta\rho_3 + \tfrac{1}{2}\theta^2\rho_2)\\ &= (0, \rho_2, \rho_3 - \theta\rho_2, \rho_4 - \theta\rho_3 + \tfrac{1}{2}\theta^2\rho_2)\ (\theta + \rho_1, 0, 0, 0).\end{aligned}$$

Soit $U'_{\lambda,\mu',\nu}$ une représentation unitaire de dimension 1 de Δ'' dont le caractère prend les valeurs $i\lambda$ en x_4, $i\mu'$ en x_3, $i\nu$ en x_2. Soit $F' \in L_{\mathbf{C}}^1(\Delta'') \cap L_{\mathbf{C}}^2(\Delta'')$. D'après la formule de Plancherel ordinaire, on a pour un bon choix de $d\delta$

$$\int_{\Delta''} |F'(\delta)|^2 d\delta = 2 \int tr(U'_{\lambda,\mu',\nu}(F')^* U'_{\lambda,\mu',\nu}(F'))d\lambda d\mu' d\nu.$$

Soit μ la valeur en $2x_2x_4 - x_3^2$ du caractère de $U'_{\lambda,\mu',\nu}$. On a $\mu = -2\lambda\mu' + \nu^2$, donc $\mu' = (\nu^2 - \mu)/2\lambda$ pour $\lambda \neq 0$. Donc

$$\int_{\Delta''} |F(\delta)|\, d\delta = 2\pi \int tr(U_{\lambda,(\nu^2-\mu)/2\lambda,\nu}(F)^* U_{\lambda,(\nu^2-\mu)/2\lambda,\nu}(F)) \left|\frac{1}{2\lambda}\right| d\lambda d\mu d\nu.$$

Maintenant, utilisons, dans **(3)**, la partie 2° de la démonstration du théorème 4. On peut y prendre $q = 2$, $a_1 = x_4$, $a_2 = 2x_2x_4 - x_3^2$, $a_3 = x_3$, $b = x_4$, $F'(\lambda, \mu) = 1/\lambda$, $R_1(\lambda, \mu) = \lambda$. On obtient la formule (14) pour un choix convenable de la mesure de Haar $d\gamma$.

Soit U une représentation unitaire irréductible de Γ_4 dont le caractère s'annule en x_4. Alors, U est triviale sur Δ, donc s'identifie à une représentation unitaire irréductible de Γ_4/Δ qui est isomorphe à Γ_3. On a $U(2x_2x_4 - x_3^2) = -U(x_3)^2$, et $U(x_3)$ est un opérateur scalaire imaginaire pur. Alors, (ii) est immédiat et (iii) résulte de la proposition 3. Enfin, (iv) est immédiat.

6. Représentations unitaires irréductibles de $\Gamma_{5,1}$. Nous utilisons la table de multiplication de $\mathfrak{g}_{5,1}$ donnée au § 1. Soient $\mathfrak{h} = \mathbf{R}x_5$ et Δ le sous-groupe correspondant de $\Gamma_{5,1}$. Il y a correspondence biunivoque entre les caractères hermitiens χ de $\mathfrak{Z}(\mathfrak{g}_{5,1})$ et les nombres réels λ. Cette correspondance est défini par la formule $\chi(x_5) = i\lambda$.

Proposition 5. (i) *Pour tout nombre réel* $\lambda \neq 0$, *il existe une représentation unitaire irréductible* U_λ *de* $\Gamma_{5,1}$ *et (à une équivalence près) une seule dont le caractère prend la valeur* $i\lambda$ *en* x_5. *La représentation* U_λ *opère dans* $L_{\mathbf{C}}^2(\mathbf{R}^2)$. *L'ensemble des vecteurs indéfiniment différentiables est* $\mathscr{S}(\mathbf{R}^2)$. *On a*

(15) $U_\lambda(x_1) = D_1$, $U_\lambda(x_2) = i\lambda M_1$, $U_\lambda(x_3) = D_2$, $U_\lambda(x_4) = i\lambda M_2$, $U_\lambda(x_5) = i\lambda$.

Si γ *est l'élément de* $\Gamma_{5,1}$ *de coordonnées* $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$, *et si* $f \in L_{\mathbf{C}}^2(\mathbf{R}^2)$, *on a*

(16) $(U_\lambda(\gamma)f)(\theta_1, \theta_2) = [\exp i\lambda(\rho_5 - \rho_2\theta_1 - \rho_4\theta_2)]f(\theta_1 + \rho_1, \theta_2 + \rho_3)$.

Si $F \in L_{\mathbf{C}}^1(\Gamma_{5,1}) \cap L_{\mathbf{C}}^2(\Gamma_{5,1})$, *on a*

$$\int_{\Gamma_{5,1}} |F(\gamma)|^2 d\gamma = \int_{\lambda\neq 0} tr(U_\lambda(F)^* U_\lambda(F))\lambda^2 d\lambda. \tag{17}$$

(ii) *Il existe une infinité de représentations unitaires irréductibles de* $\Gamma_{5,1}$, *deux à deux non équivalentes, dont le caractère prend la valeur* 0 *en* x_5. *Ce sont les représentations triviales sur* Δ; *elles s'identifient donc aux représentations unitaires de dimension* 1 *du groupe abélien* $\Gamma_{5,1}/\Delta$.

Démonstration. Soit $\mathfrak{h}' = \mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}x_2$, qui est un idéal de $\mathfrak{g}_{5,1}$ isomorphe à $\mathfrak{g}_3 \times (\mathbf{R}x_2)$. Soit Δ' le sous-groupe correspondant de $\Gamma_{5,1}$. Appliquons le lemme 24 de **(3)** à $\Gamma_{5,1}$ et Δ'; on peut prendre dans ce lemme $a_1 = x_5$, $a_2 = x_2$, $a = 1$, $b = x_5$ (compte tenu de la proposition 3). On voit que x_5 est classifiant pour $\mathfrak{g}_{5,1}$. Si $\lambda \neq 0$, il existe une représentation unitaire irréductible U_λ de $\Gamma_{5,1}$ et (à une équivalence près) une seule dont le caractère prend la valeur $i\lambda$ en x_5. Cette représentation est induite par n'importe quelle représentation unitaire irréductible de Δ' dont le caractère prend la valeur $i\lambda$ en x_5. Nous considérerons par exemple la représentation triviale sur le sous-groupe Δ'' de Δ' correspondant à $\mathbf{R}x_2$; comme Δ'/Δ'' est isomorphe à Γ_3, ceci détermine parfaitement la représentation considérée. Les formules (15) et les assertions qui les précèdent s'établissent alors en utilisant les mêmes références que pour la proposition 3. La formule (16) résulte du calcul suivant:

$$\begin{aligned}(\theta, 0, 0, 0, 0)(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) &= (\theta + \rho_1, \rho_2, \rho_3, \rho_4, \rho_5 - \theta\rho_2)\\ &= (0, \rho_2, \rho_3, \rho_4, \rho_5 - \theta\rho_2)(\theta + \rho_1, 0, 0, 0, 0).\end{aligned}$$

Utilisons, dans **(3)**, la partie 2° de la démonstration du théorème 4. D'après la proposition 3, on a $F'(\lambda) = \lambda$. D'autre part, $R_1(\lambda) = \lambda$. D'où (17). La partie (ii) de l'énoncé est évidente.

7. Représentations unitaires irréductibles de $\Gamma_{5,2}$. Nous utilisons la table de multiplication de $\mathfrak{g}_{5,2}$ donnée au § 1. Soient $\mathfrak{h} = \mathbf{R}x_4 + \mathbf{R}x_5$, et Δ le sous-groupe correspondant de $\Gamma_{5,2}$, qui est à la fois le centre et le groupe des commutateurs de $\Gamma_{5,2}$. Il y a correspondance biunivoque entre les caractères hermitiens χ de $\mathfrak{Z}(\mathfrak{g}_{5,2})$ et les systèmes (λ, μ, ν) de nombres réels. Cette correspondance est définie par les formules

$$\chi(x_4) = i\lambda, \qquad \chi(x_5) = i\mu, \qquad \chi(x_2x_5 - x_3x_4) = \nu.$$

PROPOSITION 6. *Soient* λ, μ, ν *des nombres réels.*

(i) *Si* $\lambda \neq 0$ *ou* $\mu \neq 0$, *il existe une représentation unitaire irréductible* $U_{\lambda,\mu,\nu}$ *de* $\Gamma_{5,2}$ *et (à une équivalence près) une seule dont le caractère prend les valeurs* $i\lambda$ *en* x_4, $i\mu$ *en* x_5, ν *en* $x_2x_5 - x_3x_4$. *Elle opère dans* $L_{\mathbf{C}}^2(\mathbf{R})$. *Les vecteurs indéfiniment différentiables sont ceux de* $\mathcal{S}(\mathbf{R})$. *On a*

$$U_{\lambda,\mu,\nu}(x_1) = D_1,\ U_{\lambda,\mu,\nu}(x_2) = -i\frac{\mu\nu}{\lambda^2+\mu^2} + i\lambda M_1, \tag{18}$$

$$U_{\lambda,\mu,\nu}(x_3) = i\frac{\lambda\nu}{\lambda^2+\mu^2} + i\mu M_1, \quad U_{\lambda,\mu,\nu}(x_4) = i\lambda,\ U_{\lambda,\mu,\nu}(x_5) = i\mu.$$

Si γ *est l'élément de* $\Gamma_{5,2}$ *de coordonnées* $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$, *et si* $f \in L_{\mathbf{C}}^2(\mathbf{R})$, *on a*

$$\begin{aligned}&(U_{\lambda,\mu,\nu}(\gamma)f)(\theta)\\ &\quad= \left[\exp i\left(\nu\frac{\lambda\rho_3 - \mu\rho_2}{\lambda^2+\mu^2} + \lambda(\rho_4 - \rho_2\theta) + \mu(\rho_5 - \rho_3\theta)\right)\right] f(\theta + \rho_1).\end{aligned} \tag{19}$$

Si $F \in L_{\mathbf{C}}^1(\Gamma_{5,2}) \cap L_{\mathbf{C}}^2(\Gamma_{5,2})$, *on a*

$$(20)\qquad \int_{\Gamma_{5,2}} |F(\gamma)|^2 d\gamma = \iiint_{\lambda^2+\mu^2\neq 0} tr(U_{\lambda,\mu,\nu}(F)^* U_{\lambda,\mu,\nu}(F))\, d\lambda d\mu d\nu.$$

(ii) *Si* $\lambda = \mu = 0$ *et* $\nu \neq 0$, *il n'existe aucune représentation unitaire irréductible de* $\Gamma_{5,2}$ *dont le caractère prend les valeurs* $i\lambda$ *en* x_4, $i\mu$ *en* x_5, ν *en* $x_2x_5 - x_3x_4$.

(iii) *Si* $\lambda = \mu = \nu = 0$, *il existe une infinité de représentations unitaires irréductibles de* $\Gamma_{5,2}$, *deux à deux non équivalentes, dont le caractère prend les valeurs* $i\lambda$ *en* x_4, $i\mu$ *en* x_5, ν *en* $x_2x_5 - x_3x_4$. *Ce sont les représentations triviales sur* Δ; *elles s'identifient aux représentations unitaires de dimension* 1 *du groupe abélien* $\Gamma_{5,2}/\Delta$.

Démonstration. Soient $\mathfrak{h}' = \mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}x_2$, qui est un idéal abélien de $\mathfrak{g}_{5,2}$ et Δ' le sous-groupe correspondant de $\Gamma_{5,2}$. Supposons d'abord $\mu \neq 0$. Appliquons le lemme 24 de **(3)** à $\Gamma_{5,2}$ et Δ'; on peut prendre dans ce lemme

$$a_1 = x_5,\quad a_2 = x_4,\quad a_3 = x_2x_5 - x_3x_4,\quad a_4 = x_3,\quad a = 1,\quad b = x_5.$$

On voit qu'il existe une représentation unitaire irréductible $U_{\lambda,\mu,\nu}$ de $\Gamma_{5,2}$ et (à une équivalence près) une seule dont le caractère prend les valeurs $i\lambda$ en x_4, $i\mu$ en x_5, ν en $x_2x_5 - x_3x_4$. Cette représentation est induite par n'importe quelle représentation unitaire de dimension 1 de Δ' dont le caractère prend les valeurs $i\lambda$ en x_4, $i\mu$ en x_5, ν en $x_2x_5 - x_3x_4$; nous considérerons par exemple la représentation unitaire de dimension 1 de Δ' dont le caractère prend les valeurs

$$i\lambda \text{ en } x_4,\quad i\mu \text{ en } x_5,\quad -i\frac{\mu\nu}{\lambda^2+\mu^2} \text{ en } x_2,\quad i\frac{\lambda\nu}{\lambda^2+\mu^2} \text{ en } x_3.$$

Les formules (18) et les assertions qui les précèdent s'obtiennent alors en utilisant les mêmes références que pour la proposition 3. En particulier, pour $\lambda = 0$ (et toujours $\mu \neq 0$), on a

$$U_{0,\mu,\nu}(x_1) = D_1,\quad U_{0,\mu,\nu}(x_2) = -i\frac{\nu}{\mu},\quad U_{0,\mu,\nu}(x_3) = i\mu M_1,\quad U_{0,\mu,\nu}(x_4) = 0,$$
$$U_{0,\mu,\nu}(x_5) = i\mu.$$

D'autre part, l'application linéaire de $\mathfrak{g}_{5,2}$ sur $\mathfrak{g}_{5,2}$ qui transforme x_1 en x_1, x_2 en x_3, x_3 en x_2, x_4 en x_5, x_5 en x_4, est un automorphisme de $\mathfrak{g}_{5,2}$. Il existe donc une représentation unitaire irréductible $U_{\lambda,0,\nu}$ de $\Gamma_{5,2}$ et (à une équivalence près) une seule dont le caractère prend les valeurs 0 en x_5, $i\lambda$ en x_4, ν en $x_2x_5 - x_3x_4$; cette représentation est induite par la représentation unitaire de dimension 1 de Δ' dont le caractère prend les valeurs 0 en x_5, $i\lambda$ en x_4, $i\nu/\lambda$ en x_3, 0 en x_2. Cette représentation opère dans $L_{\mathbf{C}}^2(\mathbf{R})$, les vecteurs indéfiniment différentiables sont ceux de $\mathscr{S}(\mathbf{R})$, et on a

$$U_{\lambda,0,\nu}(x_1) = D_1,\quad U_{\lambda,0,\nu}(x_2) = i\lambda M_1,\quad U_{\lambda,0,\nu}(x_3) = i\frac{\nu}{\lambda},\quad U_{\lambda,0,\nu}(x_4) = i\lambda,$$
$$U_{\lambda,0,\nu}(x_5) = 0.$$

On voit donc que les formules (18) et les assertions qui les précèdent sont valables sous la seule hypothèse que $\lambda^2 + \mu^2 \neq 0$. La formule (19) résulte du calcul suivant:

$$\begin{aligned}(\theta, 0, 0, 0, 0)(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) &= (\theta + \rho_1, \rho_2, \rho_3, \rho_4 - \rho_2\theta, \rho_5 - \rho_3\theta) \\ &= (0, \rho_2, \rho_3, \rho_4 - \rho_2\theta, \rho_5 - \rho_3\theta)(\theta + \rho_1, 0, 0, 0, 0).\end{aligned}$$

Dans **(3)**, partie 2° de la démonstration du théorème 4, $F'(\lambda, \mu, \nu) = 1/\mu$ et $R_1(\lambda, \mu, \nu) = \mu$. D'où (20). Les parties (ii) et (iii) de l'énoncé sont évidentes.

8. Représentations unitaires irréductibles de $\Gamma_{5,3}$. Nous utilisons la table de multiplication de $\mathfrak{g}_{5,3}$ donnée au § 1. Soient $\mathfrak{h} = \mathbf{R}x_5$, $\mathfrak{h}' = \mathbf{R}x_5 + \mathbf{R}x_4$, Δ et Δ' les sous-groupes correspondants de $\Gamma_{5,3}$. Alors Δ est le centre de $\Gamma_{5,3}$ et Δ' est le groupe des commutateurs de $\Gamma_{5,3}$. Il y a correspondance biunivoque entre les caractères hermitiens χ de $\mathfrak{Z}(\mathfrak{g}_{5,3})$ et les nombres réels λ. Cette correspondance est définie par la formule $\chi(x_5) = i\lambda$.

PROPOSITION 7. (i) *Pour tout nombre réel $\lambda \neq 0$, il existe une représentation unitaire irréductible U_λ de $\Gamma_{5,3}$ et (à une équivalence près) une seule dont le caractère prend la valeur $i\lambda$ en x_5. Elle opère dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$. L'ensemble des vecteurs indéfiniment différentiables est $\mathscr{S}(\mathbf{R}^2)$. On a*

$$(21)\qquad \begin{aligned}&U_\lambda(x_1) = D_1, \quad U_\lambda(x_2) = D_2 + \tfrac{1}{2}i\lambda M_1^2, \quad U_\lambda(x_3) = i\lambda M_2, \\ &U_\lambda(x_4) = i\lambda M_1, \quad U_\lambda(x_5) = i\lambda.\end{aligned}$$

Si γ est l'élément de $\Gamma_{5,3}$ de coordonnées $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$, et si $f \in L_{\mathbf{C}}^2(\mathbf{R}^2)$, on a

$$(22)\qquad (U_\lambda(\gamma)f)(\theta_1, \theta_2) = [\exp i\lambda(\rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2\theta_1^2 - \rho_3\theta_2)]f(\theta_1 + \rho_1, \theta_2 + \rho_2).$$

Si $F \in L_{\mathbf{C}}^1(\Gamma_{5,3}) \cap L_{\mathbf{C}}^2(\Gamma_{5,3})$, on a

$$(23)\qquad \int_{\Gamma_{5,3}} |F(\gamma)|^2 d\gamma = \int_{\lambda\neq 0} tr(U_\lambda(F)^* U_\lambda(F))\lambda^2 d\lambda$$

(ii) *Il existe une infinité de représentations unitaires irréductibles de $\Gamma_{5,3}$, deux à deux non équivalentes, dont le caractère prend la valeur 0 en x_5. Elles s'identifient aux représentations unitaires irréductibles du groupe $\Gamma_{5,3}/\Delta$, qui est isomorphe à $\Gamma_3 \times \mathbf{R}$. Compte tenu de la proposition 3, elles se partagent en deux séries, suivant qu'elles sont ou non triviales sur Δ'.*

Démonstration. Soient $\mathfrak{h}'' = \mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}x_2$, qui est un idéal de $\mathfrak{g}_{5,3}$, et Δ'' le sous-groupe correspondant de $\Gamma_{5,3}$. L'algèbre $\mathfrak{h}''$ est isomorphe à $(\mathbf{R}x_2 + \mathbf{R}x_3 + \mathbf{R}x_5) \times \mathbf{R}x_4$, et $\mathbf{R}x_2 + \mathbf{R}x_3 + \mathbf{R}x_5$ est isomorphe à $\mathfrak{g}_3$. Appliquons le lemme 24 de **(3)** à $\Gamma_{5,3}$ et Δ''; on peut prendre dans ce lemme $a_1 = x_5$, $a_2 = x_4$, $a = 1$, $b = x_5$. Pour $\lambda \neq 0$, il existe une représentation unitaire irréductible U_λ de $\Gamma_{5,3}$ et (à une équivalence près) une seule dont le caractère prend la valeur $i\lambda$ en x_5. Cette représentation est induite par n'importe quelle représentation unitaire irréductible de Δ'' dont le caractère prend la valeur $i\lambda$ en x_5, par exemple l'unique représentation unitaire irréductible de Δ'' dont

le caractère prend les valeurs $i\lambda$ en x_5 et 0 en x_4. Les formules (21) et les assertions qui les précèdent s'établissent alors en utilisant les mêmes références que pour la proposition 3. La formule (22) résulte du calcul suivant:

$$\begin{aligned}(\theta, 0, 0, 0, 0)(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) &= (\theta + \rho_1, \rho_2, \rho_3, \rho_4 - \theta\rho_2, \rho_5 - \theta\rho_4 + \tfrac{1}{2}\theta^2\rho_2) \\ &= (0, \rho_2, \rho_3, \rho_4 - \theta\rho_3, \rho_5 - \theta\rho_4 + \tfrac{1}{2}\theta^2\rho_2)(\theta + \rho_1, 0, 0, 0, 0).\end{aligned}$$

Avec les notations de **(3)**, partie 2° de la démonstration du théorème 4, on a $F'(\lambda) = \lambda$, $R_1(\lambda) = \lambda$. D'où (23). La partie (ii) de l'énoncé est évidente.

9. Représentations unitaires irréductibles de $\Gamma_{5,4}$. Nous utilisons la table de multiplication de $\mathfrak{g}_{5,4}$ donnée au § 1. Soient $\mathfrak{h} = \mathbf{R}x_5 + \mathbf{R}x_4$, $\mathfrak{h}' = \mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3$, Δ et Δ' les sous-groupes correspondants de $\Gamma_{5,4}$. Alors, Δ est le centre et Δ' est le groupe des commutateurs de $\Gamma_{5,4}$. Il y a correspondance biunivoque entre les caractères hermitiens χ de $\mathfrak{Z}(\mathfrak{g}_{5,4})$ et les systèmes de nombres réels λ, μ, ν. Cette correspondance est définie par les formules $\chi(x_4) = i\lambda$, $\chi(x_5) = i\mu$, $\chi(2x_1x_5 - 2x_2x_4 + x_3^2) = \nu$.

Proposition 8. *Soient λ, μ, ν des nombres réels.*

(i) *Si $\lambda \neq 0$ ou $\mu \neq 0$, il existe une représentation unitaire irréductible $U_{\lambda,\mu,\nu}$ de $\Gamma_{5,4}$ et (à une équivalence près) une seule dont le caractère prend les valeurs $i\lambda$ en x_4, $i\mu$ en x_5, ν en $2x_1x_5 - 2x_2x_4 + x_3^2$. Elle opère dans $L_{\mathbf{C}}^2(\mathbf{R})$. Les vecteurs indéfiniment différentiables sont ceux de $\mathscr{S}(\mathbf{R})$. On a*

$$\begin{aligned}&U_{\lambda,\mu,\nu}(x_1) = -\tfrac{1}{2}i\frac{\mu\nu}{\lambda^2 + \mu^2} + \frac{\lambda}{\lambda^2 + \mu^2}D_1 - \tfrac{1}{2}i\mu(\lambda^2 + \mu^2)M_1^2, \\ (24)\quad &U_{\lambda,\mu,\nu}(x_2) = \tfrac{1}{2}i\frac{\lambda\nu}{\lambda^2 + \mu^2} + \frac{\mu}{\lambda^2 + \mu^2}D_1 + \tfrac{1}{2}i\lambda(\lambda^2 + \mu^2)M_1^2, \\ &U_{\lambda,\mu,\nu}(x_3) = i(\lambda^2 + \mu^2)M_1, \quad U_{\lambda,\mu,\nu}(x_4) = i\lambda, \quad U_{\lambda,\mu,\nu}(x_5) = i\mu.\end{aligned}$$

Si γ est l'élément de $\Gamma_{5,4}$ de coordonnées $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$, et si $f \in L_{\mathbf{C}}^2(\mathbf{R})$, on a

$$\begin{aligned}&(U_{\lambda,\mu,\nu}(\gamma)f)(\theta) = \Big[\exp i\Big(-\tfrac{1}{2}\frac{\nu}{\lambda^2 + \mu^2}(\mu\rho_1 - \lambda\rho_2) + \lambda\rho_4 + \mu\rho_5 \\ (25)\quad &- \tfrac{1}{6}\frac{\mu}{\lambda^2 + \mu^2}(\lambda^2\rho_1^3 + 3\lambda\mu\rho_1^2\rho_2 + 3\mu^2\rho_1\rho_2^2 - \lambda\mu\rho_2^3) + (\lambda^2 + \mu^2)\rho_3\theta + \mu^2\rho_1\rho_2\theta \\ &+ \lambda\mu(\rho_1^2 - \rho_2^2)\theta - \tfrac{1}{2}(\lambda^2 + \mu^2)(\mu\rho_1 - \lambda\rho_2)\theta^2\Big)\Big] f\Big(\theta + \frac{\lambda\rho_1 + \mu\rho_2}{\lambda^2 + \mu^2}\Big).\end{aligned}$$

Si $F \in L_{\mathbf{C}}^1(\Gamma_{5,4}) \cap L_{\mathbf{C}}^2(\Gamma_{5,4})$, on a

$$(26)\quad \int_{\Gamma_{5,4}} |F(\gamma)|^2 d\gamma = \iiint_{\lambda^2+\mu^2\neq 0} tr(U_{\lambda,\mu,\nu}(F)^* U_{\lambda,\mu,\nu}(F))\, d\lambda d\mu d\nu.$$

(ii) *Si $\lambda = \mu = 0$ et $\nu > 0$, il n'existe aucune représentation unitaire irréductible de $\Gamma_{5,4}$ dont le caractère prend les valeurs $i\lambda$ en x_4, $i\mu$ en x_5, ν en $2x_1x_5 - 2x_2x_4 + x_3^2$.*

(iii) *Si $\lambda = \mu = 0$ et $\nu < 0$, il existe (à une équivalence près) deux représentations unitaires irréductibles de $\Gamma_{5,4}$ dont le caractère prend les valeurs $i\lambda$ en x_4,*

$i\mu$ en x_5, ν en $2x_1x_5 - 2x_2x_4 + x_3^2$. Ces représentations sont triviales sur Δ, et s'identifient à des représentations de $\Gamma_{5,4}/\Delta$, donc de Γ_3; ce sont les représentations notées $U_{\pm\sqrt{-\nu}}$ dans la proposition 3.

(iv) *Si $\lambda = \mu = \nu = 0$, il existe une infinité de représentations unitaires irréductibles de $\Gamma_{5,4}$, deux à deux non équivalentes, dont le caractère prend les valeurs $i\lambda$ en x_4, $i\mu$ en x_5, ν en $2x_1x_5 - 2x_2x_4 + x_3^2$. Ces représentations sont triviales sur Δ'. Elles s'identifient aux représentations unitaires de dimension* 1 *du groupe abélien $\Gamma_{5,4}/\Delta'$.*

Démonstration. Supposons $\lambda \neq 0$ ou $\mu \neq 0$. Soient $\mathfrak{h}'' = \mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}(\lambda x_1 + \mu x_2)$ qui est un idéal de $\mathfrak{g}_{5,4}$ de dimension 4, et Δ'' le sous-groupe correspondant de $\Gamma_{5,4}$. L'algèbre de Lie $\mathfrak{h}''$ est isomorphe à

$$(\mathbf{R}(\lambda x_1 - \mu x_2) + \mathbf{R}x_3 + \mathbf{R}(\lambda x_4 + \mu x_5)) \times \mathbf{R}(\mu x_4 - \lambda x_5),$$

et $\mathbf{R}(\lambda x_1 + \mu x_2) + \mathbf{R}x_3 + \mathbf{R}(\lambda x_4 + \mu x_5)$ est isomorphe à $\mathfrak{g}_3$, avec $\mathbf{R}(\lambda x_4 + \mu x_5)$ pour centre. Appliquons cette fois le lemme 21 de **(3)** à $\Gamma_{5,4}$ et Δ'', avec

$$x = \mu x_1 - \lambda x_2, \quad a_1 = \frac{2}{\lambda^2 + \mu^2}(\lambda x_4 + \mu x_5), \quad a = 2x_1x_5 - 2x_2x_4 + x_3^2$$

(on notera que

$$a = \frac{2}{\lambda^2 + \mu^2}((\lambda x_4 + \mu x_5)(\mu x_1 - \lambda x_2) - (\lambda x_1 + \mu x_2)(\mu x_4 - \lambda x_5)) + x_3^2).$$

La proposition 3 montre que ce qui est noté Λ dans le lemme 21 de **(3)** est l'ensemble des caractères hermitiens de $\mathfrak{Z}(\mathfrak{h}'')$ non nuls en $\lambda x_4 + \mu x_5$. Le lemme 21 (iii) de **(3)** montre que $\lambda x_4 + \mu x_5$ est classifiant pour $\mathfrak{g}_{5,4}$. En particulier, un caractère qui prend les valeurs $i\lambda$ en x_4, $i\mu$ en x_5, ν en $2x_1x_5 - 2x_2x_4 + x_3^2$ (donc la valeur $i(\lambda^2 + \mu^2) \neq 0$ en $\lambda x_4 + \mu x_5$) correspond à une représentation unitaire irréductible $U_{\lambda,\mu,\nu}$ de $\Gamma_{5,4}$ et (à une équivalence près) à une seule. D'après le lemme 21 (ii) de **(3)**, cette représentation prolonge une représentation unitaire irréductible U' de Δ'' dont le caractère prend les valeurs $i(\lambda^2 + \mu^2)$ en $\lambda x_4 + \mu x_5$ et 0 en $\mu x_4 - \lambda x_5$. D'après la proposition 3, on peut supposer que U' opère dans $L_{\mathbf{C}}^2(\mathbf{R})$, que les vecteurs indéfiniment différentiables pour U' sont les éléments de $\mathscr{S}(\mathbf{R})$, et que

$$U'(\lambda x_1 + \mu x_2) = D_1, \quad U'(x_3) = i(\lambda^2 + \mu^2)M_1,$$
$$U'(\lambda x_4 + \mu x_5) = i(\lambda^2 + \mu^2), \quad U'(\mu x_4 - \lambda x_5) = 0.$$

Alors, $U_{\lambda,\mu,\nu}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$, et, d'après le lemme 28 de **(3)**, les vecteurs indéfiniment différentiables pour $U_{\lambda,\mu,\nu}$ sont les éléments de $\mathscr{S}(\mathbf{R})$, de sorte que

$$U_{\lambda,\mu,\nu}(\lambda x_1 + \mu x_2) = D_1, \quad U_{\lambda,\mu,\nu}(x_3) = i(\lambda^2 + \mu^2)M_1,$$
$$U_{\lambda,\mu,\nu}(\lambda x_4 + \mu x_5) = i(\lambda^2 + \mu^2), \quad U_{\lambda,\mu,\nu}(\mu x_4 - \lambda x_5) = 0.$$

Naturellement, $U_{\lambda,\mu,\nu}(x_4) = i\lambda$, $U_{\lambda,\mu,\nu}(x_5) = i\mu$. Par ailleurs,

$$\begin{aligned}
\nu &= U_{\lambda,\mu,\nu}(2x_1x_5 - 2x_2x_4 + x_3^2)\\
&= \frac{2}{\lambda^2+\mu^2}[U_{\lambda,\mu,\nu}(\mu x_1 - \lambda x_2)U_{\lambda,\mu,\nu}(\lambda x_4 + \mu x_5) -\\
&\qquad U_{\lambda,\mu,\nu}(\lambda x_1 + \mu x_2)U_{\lambda,\mu,\nu}(\mu x_4 - \lambda x_5)] + U_{\lambda,\mu,\nu}(x_3)^2\\
&= \frac{2}{\lambda^2+\mu^2}\, i(\lambda^2+\mu^2)U_{\lambda,\mu,\nu}(\mu x_1 - \lambda x_2) - (\lambda^2+\mu^2)^2M_1^2\\
&= 2iU_{\lambda,\mu,\nu}(\mu x_1 - \lambda x_2) - (\lambda^2+\mu^2)^2M_1^2
\end{aligned}$$

d'où

$$U_{\lambda,\mu,\nu}(\mu x_1 - \lambda x_2) = -\tfrac{1}{2}i\nu - \tfrac{1}{2}i(\lambda^2+\mu^2)^2M_1^2.$$

On tire de là

$$U_{\lambda,\mu,\nu}(x_1) = -\tfrac{1}{2}i\frac{\nu\mu}{\lambda^2+\mu^2} + \frac{\lambda}{\lambda^2+\mu^2}D_1 - \tfrac{1}{2}i\mu(\lambda^2+\mu^2)M_1^2$$

$$U_{\lambda,\mu,\nu}(x_2) = \tfrac{1}{2}i\frac{\nu\lambda}{\lambda^2+\mu^2} + \frac{\mu}{\lambda^2+\mu^2}D_1 + \tfrac{1}{2}i\lambda(\lambda^2+\mu^2)M_1^2$$

ce qui prouve les formules (24).

Cherchons l'application exponentielle de $\mathfrak{g}_{5,4}$ dans $\Gamma_{5,4}$. La base $(x_1, x_2, x_3, x_4, x_5)$ de $\mathfrak{g}_{5,4}$ admet pour base duale dans le dual de $\mathfrak{g}_{5,4}$ le système de formes différentielles invariantes à droite sur $\Gamma_{5,4}$ obtenues au § 3. Si un élément de $\mathfrak{g}_{5,4}$ a pour coordonnées $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, le sous-groupe à 1 paramètre correspondant de $\Gamma_{5,4}$ s'obtient en cherchant les solutions du système

$$\frac{d\rho_1}{dt} = \alpha_1, \quad \frac{d\rho_2}{dt} = \alpha_2, \quad \rho_1\frac{d\rho_2}{dt} + \frac{d\rho_3}{dt} = \alpha_3,$$

$$\tfrac{1}{2}\rho_1^2\frac{d\rho_2}{dt} + \rho_1\frac{d\rho_3}{dt} + \frac{d\rho_4}{dt} = \alpha_4, \quad \rho_2\frac{d\rho_3}{dt} + \frac{d\rho_5}{dt} = \alpha_5,$$

qui s'annulent pour $t = 0$. On obtient

$$\rho_1 = \alpha_1 t, \quad \rho_2 = \alpha_2 t, \quad \rho_3 = \alpha_3 t - \tfrac{1}{2}\alpha_1\alpha_2 t^2,$$

$$\rho_4 = \alpha_4 t - \tfrac{1}{2}\alpha_1\alpha_3 t^2 + \tfrac{1}{6}\alpha_1^2\alpha_2 t^3, \quad \rho_5 = \alpha_5 t - \tfrac{1}{2}\alpha_2\alpha_3 t^2 + \tfrac{1}{3}\alpha_1\alpha_2^2 t^3.$$

En particulier, $\exp \mu_5 x_5$ est le point de coordonnées $(0, 0, 0, 0, \mu_5)$, et on a

$$U_{\lambda,\mu,\nu}(\exp \mu_5 x_5) = \exp(i\mu\mu_5);$$

$\exp \mu_4 x_4$ est le point de coordonnées $(0, 0, 0, \mu_4, 0)$, et on a

$$U_{\lambda,\mu,\nu}(\exp \mu_4 x_4) = \exp(i\lambda\mu_4);$$

$\exp \mu_3 x_3$ est le point de coordonnées $(0, 0, \mu_3, 0, 0)$, et on a

$$U_{\lambda,\mu,\nu}(\exp \mu_3 x_3) = \exp(i(\lambda^2+\mu^2)\mu_3 M_1);$$

$\exp \mu_2(\lambda x_1 + \mu x_2)$ est le point de coordonnées $(\lambda\mu_2, \mu\mu_2, -\tfrac{1}{2}\lambda\mu\mu_2^2, \tfrac{1}{6}\lambda^2\mu\mu_2^3, \tfrac{1}{3}\lambda\mu^2\mu_2^3)$, et on a

$$U_{\lambda,\mu,\nu}(\exp \mu_2(\lambda x_1 + \mu x_2)) = \exp(\mu_2 D_1);$$

$\exp \mu_1(\mu x_1 - \lambda x_2)$ est le point de coordonnées $(\mu\mu_1, -\lambda\mu_1, \frac{1}{2}\lambda\mu\mu_1^2, -\frac{1}{6}\lambda\mu^2\mu_1^3, \frac{1}{3}\lambda^2\mu\mu_1^3)$, et on a

$$U_{\lambda,\mu,\nu}(\exp \mu_1(\mu x_1 - \lambda x_2)) = \exp \mu_1(-\tfrac{1}{2}i\nu - \tfrac{1}{3}i(\lambda^2 + \mu^2)^2 M_1^2).$$

Donc

$$U_{\lambda,\mu,\nu}[\exp \mu_1(\mu x_1 - \lambda x_2) . \exp \mu_3 x_3 . \exp \mu_4 x_4 . \exp \mu_5 x_5 . \exp \mu_2(\lambda x_1 + \mu x_2)]$$
$$= \exp i(-\tfrac{1}{2}\nu\mu_1 + \lambda\mu_4 + \mu\mu_5 + (\lambda^2 + \mu^2)\mu_3 M_1 - \tfrac{1}{2}(\lambda^2 + \mu^2)^2\mu_1 M_1^2) \exp(\mu_2 D_1).$$

L'élément de $\Gamma_{5,4}$ entre crochets est

$$(\mu\mu_1, -\lambda\mu_1, \tfrac{1}{2}\lambda\mu\mu_1^2, -\tfrac{1}{6}\lambda\mu^2\mu_1^3, \tfrac{1}{3}\lambda^2\mu\mu_1^3)(0, 0, \mu_3, 0, 0)(0, 0, 0, \mu_4, 0)$$
$$(0, 0, 0, 0, \mu_5)(\lambda\mu_2, \mu\mu_2, -\tfrac{1}{2}\lambda\mu\mu_2^2, \tfrac{1}{6}\lambda^2\mu\mu_2^3, \tfrac{1}{3}\lambda\mu^2\mu_2^3)$$
$$= (\mu\mu_1, -\lambda\mu_1, \tfrac{1}{2}\lambda\mu\mu_1^2, -\tfrac{1}{6}\lambda\mu^2\mu_1^3, \tfrac{1}{3}\lambda^2\mu\mu_1^3)(\lambda\mu_2, \mu\mu_2, \mu_3 - \tfrac{1}{2}\lambda\mu\mu_2^2, \mu_4 + \tfrac{1}{6}\lambda^2\mu\mu_2^3,$$
$$\mu_5 + \tfrac{1}{3}\lambda\mu^2\mu_2^3)$$
$$= (\lambda\mu_2 + \mu\mu_1, -\lambda\mu_1 + \mu\mu_2, \mu_3 + \tfrac{1}{2}\lambda\mu(\mu_1^2 - \mu_2^2) - \mu^2\mu_1\mu_2,$$
$$-\tfrac{1}{6}\lambda\mu^2\mu_1^3 + \mu_4 + \tfrac{1}{6}\lambda^2\mu\mu_2^3 - \mu\mu_1\mu_3 + \tfrac{1}{2}\lambda\mu^2\mu_1\mu_2^2 + \tfrac{1}{2}\mu^3\mu_1^2\mu_2,$$
$$\tfrac{1}{3}\lambda^2\mu\mu_1^3 + \mu_5 + \tfrac{1}{3}\lambda\mu^2\mu_2^3 + \tfrac{1}{2}\mu^3\mu_1\mu_2^2 + \lambda\mu_1\mu_3 - \tfrac{1}{2}\lambda^2\mu\mu_1\mu_2^2 - \lambda\mu^2\mu_1^2\mu_2).$$

Posant ces coordonnées égales à $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$, on trouve, après calculs

$$\mu_1 = \frac{\mu\rho_1 - \lambda\rho_2}{\lambda^2 + \mu^2}, \qquad \mu_2 = \frac{\lambda\rho_1 + \mu\rho_2}{\lambda^2 + \mu^2},$$
$$\mu_3(\lambda^2 + \mu^2) = \rho_3(\lambda^2 + \mu^2) + \mu^2\rho_1\rho_2 + \lambda\mu(\rho_1^2 - \rho_2^2),$$
$$\lambda\mu_4 + \mu\mu_5 = \lambda\rho_4 + \mu\rho_5 - \tfrac{1}{6}\frac{\mu}{\lambda^2 + \mu^2}(\lambda^2\rho_1^3 + 3\lambda\mu\rho_1^2\rho_2 + 3\mu^2\rho_1\rho_2^2 - \lambda\mu\rho_2^3).$$

D'où la formule (25). Maintenant, appliquons à $\mathfrak{g}_{5,4}$ et à l'idéal $\mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}x_2$ de $\mathfrak{g}_{5,4}$ la partie 1° de la démonstration du théorème 4 dans **(3)**, en y faisant $a_1 = x_4$, $a_2 = x_5$, $a_3 = 2x_1x_5 - 2x_3x_4 + x_3^2$, $x = x_1$, $a = 1$, $b = 2x_5$, $F'(\lambda, \mu) = \mu$ (d'après la proposition 3), $R(\lambda, \mu) = 2\mu$. On obtient la formule (26).

Les parties (ii), (iii), (iv) de l'énoncé sont évidentes.

10. Représentations unitaires irréductibles de $\Gamma_{5,5}$. Nous utilisons la table de multiplication de $\mathfrak{g}_{5,5}$ donnée au § 1. Soient X, Y, Z, T des indéterminées. D'après le § 2, il existe un homomorphisme de $\mathbf{R}[X, Y, Z, T]$ sur $\mathfrak{Z}(\mathfrak{g}_{5,5})$ qui transforme

$$X \text{ en } x_5,\ Y \text{ en } 2x_3x_5 - x_4^2,\ Z \text{ en } 3x_2x_5^2 - 3x_3x_4x_5 + x_4^3,$$
$$T \text{ en } 9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2,$$

et dont le noyau est l'idéal de $K[X, Y, Z, T]$ engendré par $Y^3 + Z^2 - TX^2$. Si donc, pour tout caractère χ de $\mathfrak{Z}(\mathfrak{g}_{5,5})$, on pose

$$\chi(x_5) = i\lambda, \quad \chi(2x_3x_5 - x_4^2) = \mu, \quad \chi(3x_2x_5^2 - 3x_3x_4x_5 + x_4^3) = i\nu,$$
$$\chi(9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2) = \rho,$$

l'application $\chi \to (\lambda, \mu, \nu, \rho)$ est une bijection de l'ensemble des caractères de $\mathfrak{Z}(\mathfrak{g}_{5,5})$ sur l'ensemble Ξ_1 des points $(\lambda, \mu, \nu, \rho)$ de $\mathbf{C}^4$ tels que $\mu^3 + (i\nu)^2 - (i\lambda)^2\rho = 0$, c'est-à-dire tels que $\mu^3 - \nu^2 + \lambda^2\rho = 0$. Soit $\Xi = \Xi_1 \cap \mathbf{R}^4$. Si χ est hermitien, on a $(\lambda, \mu, \nu, \rho) \in \Xi$. Réciproquement, si $(\lambda, \mu, \nu, \rho) \in \Xi$, χ est changé en son conjugué par l'antiautomorphisme principal de $\mathfrak{Z}(\mathfrak{g}_{5,5})$, donc χ est hermitien. Ainsi, l'application $\chi \to (\lambda, \mu, \nu, \rho)$ définit une bijection de l'ensemble des caractères hermitiens de $\mathfrak{Z}(\mathfrak{g}_{5,5})$ sur Ξ.

Soient $\mathfrak{h} = \mathbf{R}x_5$, $\mathfrak{h}' = \mathbf{R}x_5 + \mathbf{R}x_4$, Δ et Δ' les sous-groupes correspondants de $\Gamma_{5,5}$.

PROPOSITION 9. *Soit* $(\lambda, \mu, \nu, \rho) \in \Xi$.

(i) *Si* $\lambda \neq 0$, *il existe une représentation unitaire irréductible* $U_{\lambda,\mu,\nu,\rho}$ *de* $\Gamma_{5,5}$ *et (à une équivalence près) une seule dont le caractère prend les valeurs* $i\lambda$ *en* x_5, μ *en* $2x_3x_5 - x_4^2$, $i\nu$ *en* $3x_2x_5^2 - 3x_3x_4x_5 + x_4^3$, ρ *en* $9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2$. *Elle opère dans* $L_{\mathbf{C}}^2(\mathbf{R})$. *L'ensemble des vecteurs indéfiniment différentiables est* $\mathscr{S}(\mathbf{R})$. *On a*

$$
\begin{aligned}
&U_{\lambda,\mu,\nu,\rho}(x_1) = D_1, \quad U_{\lambda,\mu,\nu,\rho}(x_2) = -\tfrac{1}{3}i\frac{\nu}{\lambda^2} - \tfrac{1}{2}i\frac{\mu}{\lambda}M_1 + \tfrac{1}{6}i\lambda M_1^3,\\
&U_{\lambda,\mu,\nu,\rho}(x_3) = -\tfrac{1}{2}i\frac{\mu}{\lambda} + \tfrac{1}{2}i\lambda M_1^2, \quad U_{\lambda,\mu,\nu,\rho}(x_4) = i\lambda M_1,\\
&U_{\lambda,\mu,\nu,\rho}(x_5) = i\lambda.
\end{aligned}
\tag{27}
$$

Si γ *est l'élément de* $\Gamma_{5,5}$ *de coordonnées* $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$ *et si* $F \in L_{\mathbf{C}}^2(\mathbf{R})$, *on a*

$$
\begin{aligned}
(U_{\lambda,\mu,\nu,\rho}(\gamma)f)(\theta) = \exp i\Big(&-\tfrac{1}{3}\frac{\nu}{\lambda^2}\rho_2 - \tfrac{1}{2}\frac{\mu}{\lambda}(\rho_3 - \rho_2\theta)\\
&+ \lambda(\rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3)\Big)f(\theta + \rho_1).
\end{aligned}
\tag{28}
$$

Si $F \in L_{\mathbf{C}}^1(\Gamma_{5,5}) \cap L_{\mathbf{C}}^2(\Gamma_{5,5})$, *on a*

$$
\begin{aligned}
&\int_{\Gamma_{5,5}} |F(\gamma)|^2 d\gamma =\\
&\qquad \iiint_{\lambda\neq 0} tr\Big(U_{\lambda,\mu,\nu,(\nu^2-\mu^3)/\lambda^2}(F)^* U_{\lambda,\mu,\nu,(\nu^2-\mu^3)/\lambda^2}(F)\Big)\frac{1}{\lambda^2}\,d\lambda\, d\mu\, d\nu.
\end{aligned}
\tag{29}
$$

(ii) *Si* $\lambda = 0$ (*donc* $\nu^2 - \mu^3 = 0$) *et* $\mu \neq 0$, *il existe une représentation unitaire irréductible* $U_{\lambda,\mu,\nu,\rho}$ *de* $\Gamma_{5,5}$ *et (à une équivalence près) une seule dont le caractère prend les valeurs* $i\lambda$ *en* x_5, μ *en* $2x_3x_5 - x_4^2$, $i\nu$ *en* $3x_2x_5^2 - 3x_3x_4x_5 + x_4^3$, ρ *en* $9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2$. *Elle opère dans* $L_{\mathbf{C}}^2(\mathbf{R})$. *L'ensemble des vecteurs indéfiniment différentiables est* $\mathscr{S}(\mathbf{R})$. *On a*

$$
\begin{aligned}
&U_{\lambda,\mu,\nu,\rho}(x_1) = D_1, \quad U_{\lambda,\mu,\nu,\rho}(x_2) = -\tfrac{1}{6}i\frac{\rho}{\nu} - \tfrac{1}{2}i\frac{\nu}{\mu}M_1^2,\\
&U_{\lambda,\mu,\nu,\rho}(x_3) = -i\frac{\nu}{\mu}M_1, \quad U_{\lambda,\mu,\nu,\rho}(x_4) = -i\frac{\nu}{\mu}, \quad U_{\lambda,\mu,\nu,\rho}(x_5) = 0.
\end{aligned}
\tag{30}
$$

Si γ *est l'élément de* $\Gamma_{5,5}$ *de coordonnées* $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$, *et si* $f \in L_{\mathbf{C}}^2(\mathbf{R})$, *on a*

$$
(U_{\lambda,\mu,\nu,\rho}(\gamma)f)(\theta) = \left[\exp i\Big(-\tfrac{1}{6}\frac{\rho}{\nu}\rho_2 - \frac{\nu}{\mu}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2)\Big)\right]f(\theta + \rho_1).
\tag{31}
$$

(iii) *Si* $\lambda = \mu = \nu = 0$, $\rho \neq 0$, *il n'existe aucune représentation unitaire irréductible de* $\Gamma_{5,5}$ *dont le caractère prend les valeurs*

$$i\lambda \text{ en } x_5,\ \mu \text{ en } 2x_3x_5 - x_4^2,\ i\nu \text{ en } 3x_2x_5^2 - 3x_3x_4x_5 + x_4^3,$$
$$\rho \text{ en } 9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2.$$

(iv) *Si* $\lambda = \mu = \nu = \rho = 0$ *il y a une infinité de représentations unitaires irréductibles de* $\Gamma_{5,5}$, *deux à deux non équivalentes, dont le caractère prend les valeurs* $i\lambda$ *en* x_5, μ *en* $2x_3x_5 - x_4^2$, $i\nu$ *en* $3x_2x_5^2 - 3x_3x_4x_5 + x_4^3$, ρ *en* $9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2$. *Ces représentations sont triviales sur* Δ', *et s'identifient aux représentations unitaires irréductibles de* $\Gamma_{5,5}/\Delta'$, *qui est isomorphe à* Γ_3.

Démonstration. Soient $\mathfrak{h}'' = \mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}x_2$, qui est un idéal abélien de $\mathfrak{g}_{5,5}$, et Δ'' le sous-groupe correspondant de $\Gamma_{5,5}$. Appliquons le lemme 24 de **(3)** à $\Gamma_{5,5}$ et Δ''; on peut prendre dans ce lemme $a_1 = x_5$, $a_2 = 2x_3x_5 - x_4^2$, $a_3 = 3x_2x_5^2 - 3x_3x_4x_5 + x_4^3$, $a_4 = x_4$, $x = x_1$, $a = 1$, $b = x_5$. Pour $\lambda \neq 0$, il existe une représentation unitaire irréductible $U_{\lambda,\mu,\nu,\rho}$ de $\Gamma_{5,5}$ et (à une équivalence près) une seule dont le caractère prend les valeurs $i\lambda$ en x_5, μ en $2x_3x_5 - x_4^2$, $i\nu$ en $3x_2x_5^2 - 3x_3x_4x_5 + x_4^3$, donc

$$\rho \text{ en } 9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2.$$

Cette représentation est induite par n'importe quelle représentation unitaire irréductible de Δ'' dont le caractère prend les valeurs $i\lambda$ en x_5, μ en $2x_3x_5 - x_4^2$, $i\nu$ en $3x_2x_5^2 - 3x_3x_4x_5 + x_4^3$, par exemple par la représentation unitaire de dimension 1 dont le caractère prend les valeurs

$$i\lambda \text{ en } x_5,\ 0 \text{ en } x_4,\ \frac{\mu}{2i\lambda} = -\tfrac{1}{2}i\frac{\mu}{\lambda} \text{ en } x_3,\ \frac{i\nu}{3(i\lambda)^2} = -\tfrac{1}{3}i\frac{\nu}{\lambda^2} \text{ en } x_2.$$

Les formules (27) et les assertions qui les précèdent s'établissent alors en utilisant les mêmes références que pour la proposition 3. La formule (28) résulte du calcul suivant:

$$\begin{aligned}
&(\theta, 0, 0, 0, 0)(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)\\
&\quad = (\theta + \rho_1, \rho_2, \rho_3 - \rho_2\theta, \rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2, \rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3)\\
&\quad = (0, \rho_2, \rho_3 - \rho_2\theta, \rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2, \rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3)\\
&\qquad (\theta + \rho_1, 0, 0, 0, 0).
\end{aligned}$$

Avec les notations de **(3)**, partie 2° de la démonstration du théorème 4, on a

$$F'(\lambda, \mu, \nu) = \frac{1}{2\lambda}\cdot\frac{1}{3\lambda^2} = \frac{1}{6\lambda^3}, \quad R_1(\lambda, \mu, \nu) = \lambda,$$

d'où la formule (29) pour un choix convenable de la mesure de Haar de $\Gamma_{5,5}$.
Si $\lambda = 0$ (donc $\nu^2 - \mu^3 = 0$), une représentation unitaire irréductible de $\Gamma_{5,5}$ dont le caractère prend la valeur $i\lambda$ en x_5 est triviale sur Δ, donc s'identifie à une représentation unitaire irréductible U' de $\Gamma_{5,5}/\Delta$; identifions $\Gamma_{5,5}/\Delta$ à

Γ_4; soient λ' et μ' les valeurs du caractère de U' en x_4 et en $2x_2x_4 - x_3^2$. Les congruences:

$$
\begin{aligned}
&2x_3x_5 - x_4^2 \equiv -x_4^2 \pmod{x_5}\\
&3x_3x_5^2 - 3x_3x_4x_5 + x_4^3 \equiv x_4^3 \pmod{x_5}\\
&9x_2^2x_5^2 - 18x_2x_3x_4x_5 + 6x_2x_4^3 + 8x_3^3x_5 - 3x_3^2x_4^2 \equiv 3x_4^2(2x_2x_4 - x_3^2) \pmod{x_5}
\end{aligned}
$$

montrent que μ, ν, ρ sont liés à λ', μ' par les relations

$$
\mu = -(i\lambda')^2, \quad i\nu = (i\lambda')^3, \quad \rho = 3(i\lambda')^2\mu'. \tag{32}
$$

Si $\mu \neq 0$ (donc $\nu \neq 0$), on en tire

$$
\lambda' = -\frac{\nu}{\mu}, \quad \frac{\mu'}{\lambda'} = \frac{\rho}{3\nu}.
$$

Les formules (30) résultent alors des formules (12) et la formule (31) de la formule (13). D'où (ii). D'autre part, (iii) résulte aussitôt des égalités (32). Enfin (iv) est immédiat.

11. Représentations unitaires irréductibles de $\Gamma_{5,6}$. Nous utilisons la table de multiplication de $\mathfrak{g}_{5,6}$ donnée au § 1. Soient $\mathfrak{h} = \mathbf{R}x_5$, Δ le sous-groupe correspondant de $\Gamma_{5,6}$. Il y a correspondance biunivoque entre les nombres réels λ et les caractères hermitiens χ de $\mathfrak{Z}(\mathfrak{g}_{5,6})$. Cette correspondance est définie par la formule $\chi(x_5) = i\lambda$.

PROPOSITION 10. (i) *Si $\lambda \neq 0$, il existe une représentation unitaire irréductible U_λ de $\Gamma_{5,6}$ et (à une équivalence près) une seule dont le caractère prend la valeur $i\lambda$ en x_5. Elle opère dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$. Les vecteurs indéfiniment différentiables sont les éléments de $\mathscr{S}(\mathbf{R}^2)$. On a*

$$
\begin{aligned}
&U_\lambda(x_1) = D_1, \quad U_\lambda(x_2) = D_2 + i\lambda M_1M_2 + \tfrac{1}{6}i\lambda M_1^3,\\
&U_\lambda(x_3) = i\lambda M_2 + \tfrac{1}{2}i\lambda M_1, \quad U_\lambda(x_4) = i\lambda M_1^2, \quad U_\lambda(x_5) = i\lambda.
\end{aligned} \tag{33}
$$

Si γ est l'élément de $\Gamma_{5,6}$ de coordonnées $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$, et si $f \in L_{\mathbf{C}}^2(\mathbf{R}^2)$, on a

$$
\begin{aligned}
(U_\lambda(\gamma)f)(\theta_1, \theta_2) = [\exp i\lambda(\rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2^2\theta_1^2 + \tfrac{1}{2}\rho_3\theta_1^2 - \tfrac{1}{6}\rho_2\theta_1^3 - \rho_3\theta_2\\
- \rho_2\theta_1\theta_2)]f(\theta_1 + \rho_1, \theta_2 + \rho_2).
\end{aligned} \tag{34}
$$

Si $F \in L_{\mathbf{C}}^1(\Gamma_{5,6}) \cap L_{\mathbf{C}}^2(\Gamma_{5,6})$, on a

$$
\int_{\Gamma_{5,6}} |F(\gamma)|^2 d\gamma = \int_{\lambda \neq 0} tr(U_\lambda(F)^* U_\lambda(F))\lambda^2 d\lambda. \tag{35}
$$

(ii) *Si $\lambda = 0$, il existe une infinité de représentations unitaires irréductibles de $\Gamma_{5,6}$, deux à deux non équivalentes, dont le caractère prend la valeur $i\lambda$ en x_5. Ces représentations sont triviales sur Δ, et s'identifient aux représentations unitaires irréductibles de $\Gamma_{5,6}/\Delta$, qui est isomorphe à Γ_4.*

Démonstration. Soient $\mathfrak{h}' = \mathbf{R}x_5 + \mathbf{R}x_4 + \mathbf{R}x_3 + \mathbf{R}x_2$, qui est un idéal de $\mathfrak{g}_{5,6}$, et Δ' le sous-groupe correspondant de $\Gamma_{5,6}$. Alors, $\mathfrak{h}'$ est isomorphe à

$(\mathbf{R}x_2 + \mathbf{R}x_3 + \mathbf{R}x_5) \times (\mathbf{R}x_4)$; et $\mathbf{R}x_2 + \mathbf{R}x_3 + \mathbf{R}x_5$ est isomorphe à $\mathfrak{g}_3$, avec $\mathbf{R}x_5$ pour centre. Les formules (33) et les assertions qui les précèdent s'obtiennent alors exactement comme pour $\Gamma_{5,3}$. La formule (34) résulte du calcul suivant:

$$
\begin{aligned}
&(\theta, 0, 0, 0, 0)(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \\
&\quad = (\theta + \rho_1, \rho_2, \rho_3 - \rho_2\theta, \rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2, \rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_2^2\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3) \\
&\quad = (0, \rho_2, \rho_3 - \rho_2\theta, \rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2, \rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_2^2\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3) \\
&\qquad\qquad (\theta + \rho_1, 0, 0, 0, 0).
\end{aligned}
$$

Avec les notations de **(3)**, partie 2° de la démonstration du théorème 4, on a $F'(\lambda) = \lambda$ et $R_1(\lambda) = \lambda$; d'où (35). La partie (ii) de l'énoncé est évidente.

12. Un autre exemple. Dans tous les exemples précédents, on a trouvé un élément classifiant a appartenant au centre de l'algèbre de Lie étudiée. Ceci permettait de considérer les représentations dont le caractère s'annule en a comme des représentations d'un groupe quotient. Nous allons voir que, malheureusement, il n'en est pas ainsi en général.

Soit $\mathfrak{g}$ l'algèbre de Lie réelle de dimension 7 dont la table de multiplication (avec les mêmes conventions qu'au § 1) est la suivante:

$$
\begin{array}{lll}
[x_1, x_2] = x_7, & [x_1, x_3] = x_6, & [x_1, x_4] = x_5, \\
[x_2, x_3] = x_5, & [x_2, x_4] = x_6, & [x_3, x_4] = x_7.
\end{array}
$$

Pour trouver $\mathfrak{J}(\mathfrak{g})$, nous avons à résoudre le système d'équations

$$
\begin{aligned}
x_7 f'_{x_2} + x_6 f'_{x_3} + x_5 f'_{x_4} &= 0, \\
- x_7 f'_{x_1} \qquad\qquad + x_5 f'_{x_3} + x_6 f'_{x_4} &= 0, \\
- x_6 f'_{x_1} - x_5 f'_{x_2} \qquad\qquad + x_7 f'_{x_4} &= 0, \\
- x_5 f'_{x_1} - x_6 f'_{x_2} - x_7 f'_{x_3} \qquad\qquad &= 0.
\end{aligned}
$$

Le déterminant de ce système par rapport aux inconnues

$$f'_{x_1}, \quad f'_{x_2}, \quad f'_{x_3}, \quad f'_{x_4}$$

n'est pas identiquement nul (il vaut $(x_5^2 + x_6^2 - x_7^2)^2$). D'où aussitôt $\mathfrak{J}(\mathfrak{g}) = \mathbf{R}[x_5, x_6, x_7]$, $\mathfrak{Z}(\mathfrak{g}) = \mathbf{R}[x_5, x_6, x_7]$.

Soit Γ le groupe de Lie simplement connexe d'algèbre de Lie $\mathfrak{g}$.

Par les mêmes méthodes que dans les paragraphes précédents, on peut montrer que $x_5^2 - x_6^2 + x_7^2$ est classifiant pour $\mathfrak{g}$. Nous n'aurons pas besoin de ce résultat; mais nous allons prouver ceci:

PROPOSITION 11. *Soit χ un caractère hermitien de $\mathfrak{Z}(\mathfrak{g})$ tel que $\chi(x_5^2 - x_6^2 + x_7^2) = 0$. Il existe une infinité de représentations unitaires irréductibles de Γ, deux à deux non équivalentes, admettant le caractère χ.*

Démonstration. Posons $\chi(x_5) = i\lambda$, $\chi(x_6) = i\mu$, $\chi(x_7) = i\nu$. On a donc $\lambda^2 - \mu^2 + \nu^2 = 0$. Distinguons trois cas:

(a) $\lambda = 0$, $\mu = \nu$. Soit $\mathfrak{h} = \mathbf{R}x_5 + \mathbf{R}(x_6 - x_7)$, qui est un idéal de $\mathfrak{g}$. Soient y_1, y_2, y_3, y_4, y_5 les images canoniques de x_1, x_2, x_3, x_4, x_6 dans $\mathfrak{g}/\mathfrak{h}$; ces éléments forment une base de $\mathfrak{g}/\mathfrak{h}$ par rapport à laquelle la table de multiplication de $\mathfrak{g}/\mathfrak{h}$ est:

$$[y_1, y_2] = y_5, \qquad [y_1, y_3] = y_5, \qquad [y_2, y_4] = y_5, \qquad [y_3, y_4] = y_5.$$

On voit que $\mathfrak{g}/\mathfrak{h}$ est isomorphe à $\mathbf{R}(y_2 - y_3) \times \mathbf{R}(y_1 + y_4) \times (\mathbf{R}y_1 + \mathbf{R}y_2 + \mathbf{R}y_5)$; et $\mathbf{R}y_1 + \mathbf{R}y_2 + \mathbf{R}y_5$ est isomorphe à $\mathfrak{g}_3$ avec $\mathbf{R}y_5$ pour centre; il existe donc une infinité de représentations unitaires irréductibles de Γ' (groupe de Lie simplement connexe d'algèbre de Lie $\mathfrak{g}/\mathfrak{h}$), deux à deux non équivalentes, dont le caractère prend une valeur donnée imaginaire pure en y_5. Par suite, il existe une infinité de représentations unitaires irréductibles de Γ, deux à deux non équivalentes, dont le caractère prend les valeurs $i\mu$ en x_6, 0 en x_5, 0 en $x_6 - x_7$. D'où la proposition dans ce cas.

(b) $\lambda = 0$, $\mu = -\nu$. La démonstration est analogue à celle qu'on vient d'utiliser dans le cas (a).

(c) $\lambda \neq 0$. Soit $\mathfrak{h}' = \mathbf{R}x_4 + \mathbf{R}x_5 + \mathbf{R}x_6 + \mathbf{R}x_7 + \mathbf{R}(\lambda x_2 - \mu x_1) + \mathbf{R}(\lambda x_3 - \nu x_1)$, qui est un idéal de $\mathfrak{g}$ de dimension 6. Soit Δ le sous-groupe de Γ correspondant à $\mathfrak{h}'$. Alors, $[\mathfrak{h}', \mathfrak{h}']$ est engendré par

$$\begin{aligned} z_1 &= [\lambda x_2 - \mu x_1, x_4] = \lambda x_6 - \mu x_5, \\ z_2 &= [\lambda x_3 - \nu x_1, x_4] = \lambda x_7 - \nu x_5, \\ z_3 &= [\lambda x_2 - \mu x_1, \lambda x_3 - \nu x_1] = \lambda(\lambda x_5 - \mu x_6 + \nu x_7). \end{aligned}$$

Soient α, β, γ des nombres réels. Soit ϕ la forme linéaire sur $\mathfrak{h}'$ telle que $\phi(\lambda x_2 - \mu x_1) = i\alpha$, $\phi(\lambda x_3 - \nu x_1) = i\beta$, $\phi(x_4) = i\gamma$, $\phi(x_5) = i\lambda$, $\phi(x_6) = i\mu$, $\phi(x_7) = i\nu$. On a

$$\begin{aligned} \phi(z_1) &= i\lambda\mu - i\lambda\mu = 0 \\ \phi(z_2) &= i\lambda\nu - i\lambda\nu = 0 \\ \phi(z_3) &= \lambda(i\lambda^2 - i\mu^2 + i\nu^2) = 0. \end{aligned}$$

Donc ϕ est nulle sur $[\mathfrak{h}', \mathfrak{h}']$. Donc il existe une représentation unitaire $U'_{\alpha,\beta,\gamma}$ de dimension 1 de Δ correspondant à ϕ. Soit $U_{\alpha,\beta,\gamma}$ la représentation unitaire de Γ induite par $U'_{\alpha,\beta,\gamma}$. En utilisant soit les méthodes des paragraphes précédents, soit le théorème 6; 2 de **(1)**, il est facile de voir que $U_{\alpha,\beta,\gamma}$ est irréductible (compte tenu du fait que $\lambda \neq 0$). D'autre part, le caractère de $U_{\alpha,\beta,\gamma}$ a même restriction à $\mathbf{R}x_5 + \mathbf{R}x_6 + \mathbf{R}x_7$ que le caractère de $U'_{\alpha,\beta,\gamma}$ **(3**, Lemme 23**)**. Le caractère de $U_{\alpha,\beta,\gamma}$ est donc χ. Pour achever la démonstration, nous allons prouver que les représentations $U_{\alpha,\beta,\gamma}$ correspondant à deux valeurs distinctes de α sont non équivalentes. Pour cela observons que

$$\begin{aligned} &[x_1, \lambda x_2 - \mu x_1 - \nu x_4] = \lambda x_7 - \nu x_5 \\ &[x_1, \lambda x_7 - \nu x_5] = 0 \\ &U'_{\alpha,\beta,\gamma}(\lambda x_2 - \mu x_1 - \nu x_4) = i(\alpha - \nu\gamma) \,.\, 1 \\ &U'_{\alpha,\beta,\gamma}(\lambda x_7 - \nu x_5) = (i\lambda\nu - i\lambda\nu) \,.\, 1 = 0. \end{aligned}$$

Il en résulte (**3**, Lemme 31) que $U_{\alpha,\beta,\gamma}\ (\lambda x_2 - \mu x_1 - \nu x_4) = i(\alpha - \nu\gamma)\ .\ 1$, ce qui prouve notre assertion.

COROLLAIRE. *Il n'existe pas, dans le centre de $\mathfrak{g}$, d'élément classifiant pour $\mathfrak{g}$.*

Démonstration. Supposons qu'il existe un élément non nul $\rho x_5 + \rho' x_6 + \rho'' x_7$ dans le centre de $\mathfrak{g}$ qui soit classifiant pour $\mathfrak{g}$. Soient λ, μ, ν, des nombres réels tels que $\rho\lambda + \rho'\mu + \rho''\nu \neq 0$, $\lambda^2 - \mu^2 + \nu^2 = 0$. Soit χ le caractère de $\mathfrak{Z}(\mathfrak{g})$ tel que $\chi(x_5) = i\lambda$, $\chi(x_6) = i\mu$, $\chi(x_7) = i\nu$. On a $\chi(\rho x_5 + \rho' x_6 + \rho'' x_7) \neq 0$, donc χ correspond à une représentation unitaire irréductible de Γ et à une seule (à une équivalence près). Mais $\chi(x_5^2 - x_6^2 + x_7^2) = 0$, ce qui contredit la proposition 11. D'où le corollaire.

13. La formule de Plancherel dans les groupes de Lie nilpotents non simplement connexes. Nous avons montré dans (**3**) que les théorèmes 1, 2, 3 de (**3**) s'étendent facilement aux groupes de Lie nilpotents non simplement connexes. Montrons maintenant que le théorème 4 de (**3**) ne se généralise pas de la même façon.

Considérons, dans Γ_4, le sous-groupe Z des éléments de coordonnées $(0, 0, 0, \rho_4)$, où ρ_4 est un entier rationnel. Ce sous-groupe est discret central. Soit $\Gamma_4' = \Gamma_4/Z$. Les représentations unitaires irréductibles de Γ_4' s'identifient aux représentations unitaires irréductibles de Γ_4 dont le caractère prend en x_4 des valeurs de la forme $2i\pi\tau$, avec τ entier rationnel. Celles de ces représentations qui sont déterminées (à une équivalence près) par leur caractère sont les représentations notées, dans la proposition 4, $U_{2\pi\tau,\mu}$ avec τ entier rationnel $\neq 0$.

Or, nous allons voir que, pour toute fonction intégrable F sur Γ_4' constante sur les classes suivant le centre (compact) de Γ_4', et pour tout entier rationnel $\tau \neq 0$, on a $U_{2\pi\tau,\mu}(F) = 0$. Ceci prouvera qu'une "formule de Plancherel" pour Γ_4' doit faire intervenir des représentations unitaires irréductibles de Γ_4' non déterminées par leurs caractères.

Comme coordonnées sur Γ_4', nous utiliserons les coordonnées $\rho_1, \rho_2, \rho_3, \rho_4$ déduites des coordonnées de même nom sur Γ_4 par passage au quotient: ρ_1, ρ_2, ρ_3 sont des nombres réels, et ρ_4 est un nombre réel modulo 1. Si $f \in L_{\mathbf{C}}^2(\mathbf{R})$ on a, d'après (13), et en posant $\lambda = 2\pi\tau$:

$$(U_{2\pi\tau,\mu}(F)f)(\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^1 F(\rho_1, \rho_2, \rho_3)$$
$$\left[\exp i\left(-\tfrac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2\right)\right]f(\theta + \rho_1)d\rho_1 d\rho_2 d\rho_3 d\rho_4$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\rho_1, \rho_2, \rho_3)\left[\exp i\left(-\tfrac{1}{2}\frac{\mu}{\lambda}\rho_2 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2\right)\right]$$
$$f(\theta + \rho_1)d\rho_1 d\rho_2 d\rho_3 \int_0^1 (\exp i\lambda\rho_4)d\rho_4.$$

Comme τ est entier $\neq 0$, on a

$$\int_0^1 (\exp 2i\pi\tau\rho_4)d\rho_4 = 0,$$

donc

$$(U_{2\pi\tau,\mu}(F)f)(\theta) = 0.$$

D'où notre assertion.

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THE BETTI NUMBERS OF THE SIMPLE LIE GROUPS

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1. Introduction. The purpose of the present paper[1] is to simplify the calculation of the Betti numbers of the simple compact Lie groups.

For the unimodular group and the orthogonal group on a space of odd dimension the form of the Poincaré polynomial was correctly guessed by E. Cartan in 1929 (**5**, p. 183). The proof of his conjecture and its extension to the four classes of classical groups was given by L. Pontrjagin (**13**) using topological arguments and then by R. Brauer (**2**) using algebraic methods. However, the case of the exceptional Lie groups proved more recalcitrant and was finally settled only in 1949 by C. T. Yen (**21**). Borel and Chevalley (**1**) have recently simplified the calculations for the exceptional groups. Even so, they make use of a large number of disparate algebraic and topological results including the known facts for the classical groups. Much of their paper was already covered by results of Coxeter (**9**) and Racah (**14**). Their method entails a tedious discussion of special cases.

Hopf (**10**) and Samelson (**15**) showed that for a compact Lie group the Poincaré polynomial (the coefficients of which are the Betti numbers) is of the form

$$P(t) = \prod_{i=1}^{n} (1 + t^{p_i})$$

where n is the rank of the group and p_i are odd integers. Chevalley (**6**) proved that $p_i = 2k_i - 1$ where k_i is the degree of a minimal homogeneous invariant of the group. We shall show how the k_i may be easily obtained.

As is well known (**18**), the classification of simple compact Lie groups of rank n is closely related to the classification of finite orthogonal groups generated by reflections in a space of dimension n. Since this finite group associated with the Lie group was first introduced by Killing, we shall call it the Killing group and denote it by $\mathfrak{K}$. This group has also been called the "kaleidoscopic group" and the "Weyl group." The latter name has little justification since, though Weyl made use of it in 1926, it was used by Killing and, following him, Cartan, to effect the classification of simple Lie groups. All particular cases of the Killing group were described in some detail in Cartan's paper (**4**) in 1896. It has been studied and used by many authors including Cartan (**3**, p. 58; **4**), Coxeter (**8**, chap. 11), Stiefel (**18**), Weyl

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(**19**, § 4 et seq.), and Witt (**20**). The Killing group of a compact Lie group is isomorphic to the quotient group relative to $\mathfrak{T}$ of the normalizer of a maximal abelian subgroup $\mathfrak{T}$. Chevalley (**6**) has shown that by restricting the minimal invariants to $\mathfrak{T}$, the problem of finding k_i reduces to that of finding the degrees of the minimal homogeneous invariants of the Killing group represented by orthogonal transformations in a space of n dimensions. Here and throughout the paper n is the rank of the Lie group.

2. The product of reflections. For a simple Lie group, $\mathfrak{G}$, the Killing group, $\mathfrak{K}$, represented as a group of congruent transformations in Euclidean n-space, E_n, is generated by n reflections which we shall denote by R_i, $1 \leqslant i \leqslant n$. The relations among R_i are conveniently indicated by a Coxeter graph (**8**, § 11.3, and p. 297). The Coxeter graph of $\mathfrak{G}$ consists of n nodes joined by branches. The ith node corresponds to R_i. Two nodes are joined if the corresponding two reflections do not commute. The (i, j) branch is marked to indicate the period of R_iR_j, but we can neglect this for present purposes. We shall, however, distinguish two types of Coxeter graph.

In *Case* I the graph consists of a single chain such as

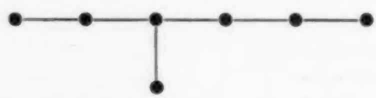
which is the graph for A_5. Such graphs occur for the following simple Lie groups:

A_n (unimodular), B_n (orthogonal group on $2n + 1$ variables),

C_n (symplectic), G_2 (the exceptional group of rank 2 and dimension 14),

F_4 (the exceptional group of rank 4 and dimension 52).

In *Case* II the graph consists of a principal chain with $n - 1$ nodes, with a second chain containing one node emanating from the principal chain. For example,



is the graph for E_7. *Case* II includes:

D_n (the orthogonal group on $2n$ variables),

E_6, E_7, E_8 (the exceptional groups of rank 6, 7, 8 and dimension 78, 133, 248 respectively).

The product of the n generating reflections, $R = R_1R_2 \ldots R_n$ and its order, h, play a fundamental role in what follows. It is of historical interest to note that Killing (**12**, pp. 18-23; **3**, p. 58) made use of this same product. Coxeter has made a careful study of R (**8**, § 12.3; **9**) and we depend heavily on his work. In particular he noticed that if ζ is a primitive hth root of unity and

$$\zeta^{m_i}, \quad \text{where} \quad 0 < m_i < h,$$

are the eigenvalues of R, then $k_i = m_i + 1$. We shall prove this. Hence Coxeter's calculation of m_i determines k_i. We shall call the positive integers *m_i the exponents of $\mathfrak{K}$*.

Definition. By a ***regular vector*** we shall mean one which does not lie on a reflecting hyperplane of $\mathfrak{R}$. Thus $\mathbf{x}$ is regular if and only if $\mathbf{r}_a \cdot \mathbf{x} \neq 0$ for every positive root vector $\mathbf{r}_a$, since the root vectors are orthogonal to the reflecting hyperplanes of $\mathfrak{R}$.

LEMMA 1. *The operation R, of order h, has a primitive* hth *root of unity, ζ, as an eigenvalue which corresponds to a regular eigenvector. With a_{ij} as defined below and α the minimum eigenvalue of (a_{ij}), $\zeta = e^{i\theta}$, where $\theta = 2\pi/h =$* $4 \arcsin (\frac{1}{4}\alpha)^{\frac{1}{2}}$.

Proof. As was shown by Cartan (**3**, p. 58) and by Coxeter (**9**, p. 767) the equation

$$R\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

is equivalent to the equations

$$b_i^j x_j = 0 \tag{2}$$

where the co-ordinates x_j are distances from the sides of a fundamental simplex $\mathbf{F}$ of $\mathfrak{R}$. If $\mathbf{e}_i$ are unit vectors orthogonal to sides of $\mathbf{F}$, pointing inwards, and $\mathbf{e}^i$ is the reciprocal basis of E_n such that $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$, then $\mathbf{x} = x_i \mathbf{e}^i$. The matrix b_i^j has the form:

$$\begin{pmatrix} \frac{1}{2}(\lambda + 1) & a_{12}\lambda & \cdots & a_{1n}\lambda \\ a_{21} & \frac{1}{2}(\lambda + 1) & \cdots & a_{2n}\lambda \\ & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & \frac{1}{2}(\lambda + 1) \end{pmatrix}$$

where $a_{ij} = a_{ji} = -\cos(\pi/p_{ij})$, and p_{ij} is the period of R_iR_j. Thus $a_{ii} = 1$, $a_{ij} < 0$ if the distinct i and j nodes are connected by a branch, and $a_{ij} = 0$ otherwise. The matrix (a_{ij}) therefore corresponds to what Coxeter calls an a-form (**8**, § 10.2). We order the co-ordinates $x_1, x_2, \ldots, x_{n-1}$ to correspond in succession to the nodes of the principal chain from left to right, with x_n corresponding to the end node in Case I, and to the node on the second chain in Case II. In Case II we let $q = 2$ for D_n and $q = 3$ for E_6, E_7, E_8.

With the notation fixed in this way the only non-vanishing elements in the ith row of (a_{ij}) are:

(a) *in Case* I: for $i = 1$, a_{11}, a_{12}; for $i = n$, a_{nn-1}, a_{nn}; otherwise a_{ii-1}, a_{ii}, a_{ii+1};

(b) *in Case* II: for $i = 1$, a_{11}, a_{12}; for $i = q$, $a_{qq-1}, a_{qq}, a_{qq+1}, a_{qn}$; for $i = n - 1$, $a_{n-1\,n-2}, a_{n-1\,n-1}$; for $i = n$, a_{nq}, a_{nn}; otherwise, $a_{ii-1}, a_{ii}, a_{ii+1}$.

The equations (2) are now transformed in *Case* I by setting

$$x_j = \lambda^{-\frac{1}{2}j} y^j \tag{3a}$$

and multiplying the ith row by $\lambda^{\frac{1}{2}(i-1)}$; and in *Case* II by setting

$$x_j = \lambda^{-\frac{1}{2}j} y^j, \qquad 1 < j < n - 1; \qquad x_n = \lambda^{-\frac{1}{2}(q+1)} y^n \tag{3b}$$

and multiplying the ith row by $\lambda^{\frac{1}{2}(i-1)}$ for $1 \leq i \leq n-1$, and the nth row by $\lambda^{\frac{1}{2}r}$. Equations (2) then take the form

$$(a_{ij} - (1 - \Lambda)\delta_{ij})y^j = 0 \tag{4}$$

where

$$\Lambda = \tfrac{1}{2}(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}).$$

For $\lambda = 1$, (2) reduces to $a_{ij}y^j = 0$ which has no non-trivial solution for the simple groups. Thus no vector is fixed under R, and $-1 < \Lambda < 1$.

Let α be the smallest eigenvalue of (a_{ij}), then $(a_{ij} - \alpha\delta_{ij})y^iy^j$ is a positive semi-definite connected a-form, so that $(a_{ij} - \alpha\delta_{ij})$ is of nullity 1 and the equations (4), with $\alpha = 1 - \Lambda$ have a solution y_0^j with $y_0^j > 0$ for all j. Let $\zeta = e^{i\theta}$ be the corresponding value of λ where $0 < \theta < 2\pi$. Since α is the minimum eigenvalue of (a_{ij}), θ is the smallest positive angle ϕ for which $e^{i\phi}$ is an eigenvalue of R. The angle $\theta = 2\pi/p$ where p is an integer. For let p be the smallest integer such that $p\theta$ is a multiple of 2π. Then $e^{i\theta}$ is a primitive pth root of unity. Since the characteristic equation of R has rational coefficients it has as roots all primitive pth roots of unity and in particular $e^{i2\pi/p}$. The above minimum property of θ implies that $\theta = 2\pi/p$.

Equations (3a) or (3b), with λ replaced by ζ and y^j by y_0^j, give an eigenvector $\mathbf{x}_0$ of R. We must show that $\mathbf{r}_\alpha \cdot \mathbf{x}_0 \neq 0$ for all positive root vectors $\mathbf{r}_\alpha$. To this end, note that $\theta \leq 2\pi/(n+1)$. We prove this by induction on n. For $n = 1$, $R = R_1$, $\lambda = -1$, $\theta = 2\pi/2$. In general, the minimum eigenvalue α of (a_{ij}), is less than or equal to the minimum eigenvalue α' of the sub-matrix $(a_{ij})'$ with $i, j > 1$, corresponding to the Coxeter graph obtained by removing the first node of the given graph. Now y_0^1, which does not vanish, is proportional by (**8**, 10.27) to the square-root of the (1, 1) principal minor of $(a_{ij} - \alpha\delta_{ij})$. Thus $(a_{ij} - \alpha\delta_{ij})'$ is regular and α is not an eigenvalue of $(a_{ij})'$. Thus α is in fact less than α' and if $\theta' = 2\pi/t$, $\theta \leq 2\pi/(t+1)$. This enables us to complete the induction.

The fact that $\mathbf{x}_0$ is regular now follows easily. Each positive root vector $\mathbf{r}_\alpha = r_\alpha^j\mathbf{e}_j$ where r_α^j are positive rational numbers. Thus

$$\mathbf{r}_\alpha \cdot \mathbf{x}_0 = r_\alpha^j x_j = \sum_j (r_\alpha^j \cos \phi_j y_0^j - i r_\alpha^j \sin \phi_j y_0^j)$$

where the ϕ_j are obtained from ζ by equations (3): in *Case* I,

$$0 < \phi_j < \tfrac{1}{2}j\theta \leq \pi\frac{j}{n+1} < \pi$$

and similarly in *Case* II, $0 < \phi_j < \pi$. Thus $\sin \phi_j > 0$ and $\Sigma r_\alpha^j \sin \phi_j y_0^j > 0$ which implies $\mathbf{r}_\alpha \cdot \mathbf{x}_0 \neq 0$.

To show that ζ is a primitive hth root, assume that $\zeta^u = 1$, for $u \leq h$. Then $R^u\mathbf{x}_0 = \mathbf{x}_0$. But $\mathbf{x}_0$ does not lie on a reflecting plane, so it is fixed only under the identity (**19**). Hence $R^u = I$, and $u = h$.

Since $\Lambda = 1 - \alpha = \cos \frac{1}{2}\theta = 1 - 2\sin^2\frac{1}{4}\theta$, we have $\theta = 4 \text{ arc sin } (\frac{1}{2}\alpha)^{\frac{1}{2}}$.

COROLLARY 1. *All the primitive hth-roots of unity occur as eigenvalues of R and the corresponding eigenvectors are regular.*

Proof. Since the characteristic equation of R has rational coefficients, the first part of the statement follows immediately.

The mapping of ζ onto ζ', another primitive h-root of unity, while keeping the rationals $\mathscr{R}$ fixed, determines an automorphism of the field $\mathscr{R}(\zeta)$, which sends the co-ordinates x_{0i} of $\mathbf{x}_0$ into the co-ordinates of the eigenvector $\mathbf{x}_0'$ corresponding to ζ'. Under this mapping $\mathbf{r}_\alpha \cdot \mathbf{x}_0$, which is different from zero will not be mapped onto zero. Therefore $\mathbf{x}_0'$ is regular.

It follows from this that $\phi(h) \leqslant n$, where ϕ is Euler's function. From the proof of the theorem we also have the limitation $h \geqslant n + 1$.

COROLLARY 2. *The number of reflections in $\mathfrak{R}$ is an integral multiple of $\frac{1}{2}h$.*

Proof. $R^u\mathbf{r}_\alpha \cdot \mathbf{x}_0 = \mathbf{r}_\alpha \cdot R^{-u}\mathbf{x}_0 = \zeta^{-u}(\mathbf{r}_\alpha \cdot \mathbf{x}_0)$. These are distinct and different from zero for $1 \leqslant u \leqslant h$. Thus $R^u\mathbf{r}_\alpha$ are distinct. The desired result follows, if we note that $\mathbf{r}_\alpha$ and $-\mathbf{r}_\alpha$ give rise to the same reflection, by partitioning the reflections into equivalent classes under the cyclic group generated by R.

It is in fact easy to verify **(8**, 12.61**)** that *the number of reflections is equal to*

$$\tfrac{1}{2}nh. \tag{5}$$

3. The Jacobian of a basic set of invariants. Chevalley **(7)** has given an elegant proof of the fact that any polynomial in x which is invariant under $\mathfrak{R}$ belongs to the ring generated by n minimal invariants I_i. If I_i has degree k_i then by a theorem of Molien **(16)**,

$$g \prod_{i=1}^{n} (1 - t^{k_i})^{-1} = \sum_k \prod_{i=1}^{n} (1 - \omega_i^k t)^{-1} \tag{6}$$

where g is the order of $\mathfrak{R}$ and ω_i^k are the eigenvalues of the operator $k \in \mathfrak{R}$. Following Shephard and Todd **(16**, p. 289**)** we multiply (6) by $(1 - t)^n$ and set $t = 1$, whence $g = \Pi k_i$. Subtract $(1 - t)^{-n}$ from both sides of (6), multiply by $(1 - t)^{n-1}$, set $t = 1$ and we deduce that *the number of reflections in $\mathfrak{R}$* is

$$\Sigma(k_i - 1). \tag{7}$$

Consider the equations $I_i(x) = w_i$, where I_i are any n algebraically independent polynomial invariants of $\mathfrak{R}$. For a point $\mathbf{x}$ at which the Jacobian $J = |\partial_j I_i| \neq 0$, there will be open neighbourhoods of $\mathbf{x}$ and $\mathbf{w}$ in one-to-one correspondence. However, if $\mathbf{x}$ lies on one of the reflecting hyperplanes of $\mathfrak{R}$, any open neighbourhood of $\mathbf{x}$ contains points which are equivalent under $\mathfrak{R}$ and correspond to the same point $\mathbf{w}$. Thus $J = 0$ on the reflecting hyperplanes of $\mathfrak{R}$.

In particular, if I_i are a set of minimal invariants the degree $\Sigma(k_i - 1)$ of J is equal to the number of reflecting hyperplanes of $\mathfrak{R}$. Hence, **(17)**.

LEMMA 2. *The Jacobian of n minimal polynomial invariants of $\mathfrak{R}$ is equal, within a multiplicative constant, to the product of the linear forms whose vanishing gives the reflecting hyperplanes of $\mathfrak{R}$.*

With the above preparation, we may now easily reach the main result of the paper by evaluating J for a set of n minimal invariants, in a system of co-ordinates in which R is diagonal. Let $\mathbf{u}_i$ be an eigenvector of R such that

$$R\,\mathbf{u}_i = \zeta^{m_i}\,\mathbf{u}_i$$

where m_i are the exponents of $\mathfrak{R}$, with $m_1 = 1$. Thus if $\mathbf{x} = x^i\mathbf{u}_i$ goes into

$$\bar{\mathbf{x}} = R\mathbf{x} = x^i\zeta^{m_i}\mathbf{u}_i = \bar{x}^i\mathbf{u}_i$$

then

$$\bar{x}^i = \zeta^{m_i}\,x^i.$$

Since $\mathbf{u}_1$ is regular by Lemma 1, $J \neq 0$ at $\mathbf{x} = x^1\mathbf{u}_1$. Thus for each i there is a j such that $\partial_j I_i \neq 0$ at $x^1\mathbf{u}_1$. But at this point $\partial_j I_i$ is a multiple of

$$(x^1)^{k_i-1}$$

and this term arises from a term

$$(x^1)^{k_i-1}x^j$$

in I_i. This term must be invariant under R; therefore

$$k_i - 1 + m_j \equiv 0 \ (h). \tag{8}$$

Since R is real, together with m_j, $h - m_j$ is an exponent, and since for $J \neq 0$ all j must occur, by reordering m_j we can arrange that $k_i \equiv m_i + 1$ (h). But

$$\sum m_j = \sum \ (h - m_j) = \tfrac{1}{2}nh = \sum \ (k_i - 1)$$

by (5). Hence

$$k_i = m_i + 1.$$

This conclude the proof[2] of the

THEOREM. *The degrees k_i of a set of minimal polynomial invariants of the Killing group, $\mathfrak{R}$, are given by $k_i = m_i + 1$, where m_i are the exponents of $\mathfrak{R}$.*

Hence from Coxeter's elegant calculation of m_i (9) we obtain the $p_i = 2k_i - 1 = 2m_i + 1$, which define the Poincaré polynomial. For the simple compact Lie groups the p_i are as follows

A_n	:	$3, 5, 7, 9, \ldots, 2n + 1$
B_n, C_n	:	$3, 7, 11, 15, \ldots, 4n - 1$
D_n	:	$3, 7, 11, \ldots, 4n - 5, 2n - 1$
G_2	:	$3, 11$
F_4	:	$3, 11, 15, 23$
E_6	:	$3, 9, 11, 15, 17, 23$
E_7	:	$3, 11, 15, 19, 23, 27, 35$
E_8	:	$3, 15, 23, 27, 35, 39, 47, 59.$

[2]Professor Coxeter has pointed out that the proof of this theorem is valid not only for the groups $\mathfrak{R}$ associated with Lie Groups but for any real finite group generated by reflections.

4. Remarks.

(i) If h is known, Coxeter's calculation of m_i can sometimes be simplified. The primitive h-roots of unity are given by ζ^u where $(u, h) = 1$. For E_8, $h = 30$ and the possible u are 1, 7, 11, 13, 17, 19, 23, 29 giving us the eight m_i. For E_7, $h = 18$, giving 1, 5, 7, 11, 13, 17 for m_i. The seventh root of unity must be real, therefore equal to -1 corresponding to $m_i = 9$. For E_6, $h = 12$ and the above method determines only four of the m_i : 1, 5, 7, 11 and further argument is needed to obtain 4 and 8. For F_4, $h = 12$ giving 1, 5, 7, 11.

From the form of the equations (2) one easily proves they are of nullity one except in Case II for $\lambda = -1$ when they have nullity two. Hence all the eigenvalues are simple except $\lambda = -1$ in Case II which is double. For A_n, $h = n + 1$. We know that $\lambda = 1$ is not an eigenvalue, and that the eigenvalues are all different; so they are completely determined.

(ii) The poles on each side of (6) coincide, therefore if any element of $\mathfrak{R}$ has as eigenvalue a primitive pth root of unity then p divides k_i for some i. Since each subgroup of $\mathfrak{G}$ has associated with it a primitive root of unity by Lemma 1, the k_i provide a limitation on the possible subgroups. Conversely, a knowledge of subgroups partially determines k_i. This, evidently, is the basis of many of the topological arguments for determining the Betti numbers by discussion of subgroups.

(iii) The symmetry in the sequence of first differences of the p_i sometimes referred to as "double duality" is explained by the simple fact that R is a real operator and together with λ, $\bar{\lambda}$ is an eigenvalue.

(iv) Previous methods of obtaining k_i depended on the explicit construction of a set of minimal invariants. These are partly determined by our method. For if $m_s + m_{\bar{s}} = h$ the invariant I_s of degree $k_s = m_s + 1$ contains the term

$$x_1^{m_s} x_{\bar{s}},$$

Indeed, there will be a term in I_s of the form

$$x_i^{m_s} x_j.$$

where $m_s m_i + m_j \equiv 0(h)$, for each m_i relatively prime to h. Probably

$$I_s = \sum_k (kx_1)^{m_s} (kx_{\bar{s}})$$

where k ranges over a set of representatives of the cosets of $\mathfrak{R}$ with respect to the cyclic group R, but this has not been proved.

(v) The above proof still contains the inelegancy of using (5) which has hitherto only been proved by verification. This is unsatisfactory even if one admits that once the h are known the verification is trivial. It would be most desirable to give a general proof[3] of the following three facts, perhaps not

[3]Since this was written, I have learned that R. Steinberg has a paper in the course of publication which deals with (a) and (b).

unrelated, which have been observed: (a) the number of reflections in $\mathfrak{R}$ is $\frac{1}{2}nh$; (b) if the dominant root vector is $z^i\mathbf{t}_i$, where $\mathbf{t}_i$ are a basic set of simple roots, then $h = 1 + \Sigma z^i$; (c) $\mathfrak{R}$ contains a subgroup isomorphic to $\mathfrak{S}_n$, the symmetric group on n objects.

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RETRACEABLE SETS

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1. Introduction. Let us compare two properties of sets of non-negative integers: (1) the set α has *property* Γ, if there exists an effective procedure which when applied to any element of α different from its maximum (which α does not necessarily possess) yields the next larger element of α; (2) the set α has *property* Δ, if there exists an effective procedure which when applied to any element of α different from its minimum yields the next smaller element of α. It is readily seen that every recursive set has both properties. Let α be any infinite set with property Γ. We define: $f(0)$ = the minimum of α, $f(n+1)$ = the element obtained when the effective procedure is applied to $f(n)$. It is clear that $f(n)$ is a strictly increasing recursive function ranging over α. The class of all sets with property Γ is therefore the same as the well-known denumerable class of all recursive sets. We now show that there are c non-recursive sets which possess property Δ. Let $\{a_n\}$ be any sequence of numbers chosen from the set $(0, \ldots, 9)$, but such that $a_0 \neq 0$. Put

$$\sigma = (a_0, 10 \cdot a_0 + a_1, 10^2 \cdot a_0 + 10 \cdot a_1 + a_2, \ldots)$$

$$f(x) = \begin{cases} x \text{ for } 0 < x < 9, \\ \left[\dfrac{x}{10}\right] \text{ for } x > 10. \end{cases}$$

We see that $f(x)$ is a recursive function which maps a_0 onto itself and every other element of σ onto the next smaller element of σ. Thus, σ is an infinite set with property Δ. The sequence $\{a_n\}$ can be chosen in c different ways and different choices of $\{a_n\}$ yield different sets σ. Since there are only c sets (that is, of non-negative integers), it follows that there exist exactly c sets with property Δ; only $\aleph_0$ of these are recursive, hence c are non-recursive.

It is the purpose of this paper to prove several theorems concerning sets with property Δ (henceforth called *retraceable sets*), in particular:

(1) *every degree of unsolvability can be represented by a retraceable set;*

(2) *every degree of unsolvability which can be represented by a r.e.* (*that is, recursively enumerable*) *set can also be represented by a r.e. set with a retraceable complement.*

2. Notations and terminology. A non-negative integer is called a *number*, a collection of numbers is called a *set* and a collection of sets a *class.*

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The Boolean operations of union, intersection and complementation are written $+$, $\cdot$, $'$ respectively; $\subset$ stands for inclusion (proper or not), ϕ for the empty set of numbers, ϵ for the set of all numbers, δf and ρf for the domain and range of the function $f(x)$ respectively, and $fg(x)$ for the function $f(g(x))$. The cardinal of a collection Θ is denoted by card Θ, the minimum of α (in case $\alpha \neq \phi$) by min α and the maximum of α (in case $\alpha \neq \phi$ and is finite) by max α. We write $c_\sigma(x)$ for the characteristic function of σ and $p_\sigma(x)$ for the function with domain σ which maps the minimum of σ onto itself and every other element of σ onto the next smaller element of σ. In the special case $\sigma = \phi$ the function $p_\sigma(x)$ is nowhere defined; if $\sigma = \epsilon$ it is the usual predecessor function. If α is an infinite set, the strictly increasing function ranging over α is called the ***principal function of*** α and denoted by $h_\alpha(x)$. The function $f(x)$ is ***downward*** if $f(x) \leqslant x$ for every $x \in \delta f$. The sets α and β are ***separable*** (written $\alpha|\beta$) if there are disjoint r.e. sets α_1 and β_1 such that $\alpha \subset \alpha_1$ and $\beta \subset \beta_1$.

We assume the reader to be familiar with the following notions: array, discrete array, immune set, simple set, hypersimple set (see, for instance (**1**)). A set is ***hyperimmune*** if it is infinite and its complement includes at least one row of every discrete array. We shall use the fact due to Rice (**7**, Theorem 21) that the infinite set α is hyperimmune if and only if $h_\alpha(x)$ is not bounded by any recursive function. It is well-known that there is an array in which every finite set occurs exactly once (**6**, p. 304); $\{\rho_n\}$ will denote a specific array of this type which has the additional property $\rho_0 = \phi$.

Definition. The set α is ***retraceable***, if $p_\alpha(x)$ has a partial recursive extension. If α is retraceable, every partial recursive extension of $p_\alpha(x)$ is a ***retracing function of*** α or a ***function which retraces*** α. A function $r(x)$ is a ***retracing function*** if it retraces at least one infinite set.

Definition. The number a is an ***initial number*** of the downward function $f(x)$ if $a \in \delta f$ and $f(a) = a$. The set of all initial numbers of a downward function is the ***initial set*** of that function.

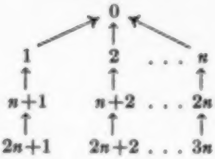
Definition. A set is ***introreducible***, if it is (Turing) reducible to each of its infinite subsets.

The notions of retraceability and introreducibility were communicated by R. S. Tennenbaum.

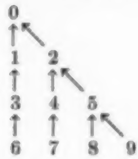
3. Examples. Many downward functions are conveniently described by a diagram. Let n be any number $\geqslant 1$, and let $p_0, p_1, \ldots$ be the sequence of all primes arranged according to size. The following six diagrams (supposed to be extended indefinitely) are self-explanatory.

We denote the functions described in the six diagrams by $r_1(x), \ldots, r_6(x)$ respectively. Each of the functions $r_1(x), \ldots, r_5(x)$ is partial recursive and retraces at least one infinite set; $r_6(x)$ is partial recursive and downward, but retraces only finite sets. For $1 \leqslant i \leqslant 6$, let σ_i stand for the initial set of $r_i(x)$

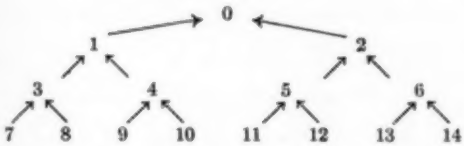
EXAMPLE 1



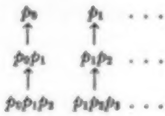
EXAMPLE 2



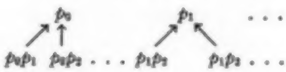
EXAMPLE 3



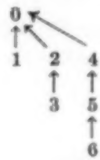
EXAMPLE 4



EXAMPLE 5



EXAMPLE 6



and T_i for the class of all *infinite* sets retraced by $r_i(x)$. Knowing the cardinality of σ_i does not tell us much about the cardinality of T_i or vice versa. For the cardinalities of $\sigma_1, \ldots, \sigma_6$ are 1, 1, 1, $\aleph_0$, $\aleph_0$, 1 respectively and those of $T_1, \ldots, T_6$ are n, $\aleph_0$, c, $\aleph_0$, c, 0 respectively. Note that $r_2(x)$ and $r_4(x)$ have the common property that they retrace exactly denumerably many infinite sets, all of which are recursive; but, while the sets in T_4 are mutually disjoint, the sets in T_2 all contain 0, that is, the only initial number of $r_2(x)$. Examples 3 and 5 illustrate the existence of retracing functions which retrace c mutually almost disjoint infinite sets.

4. Propositions. Let α be an immune, retraceable set and $f(x)$ a retracing function of α. Then $\alpha \subset \delta f$ and this inclusion must be proper, because δf is r.e. In certain cases we can actually find elements in δf which cannot belong to α. For if $f(x)$ is not downward every element x_1 such that

$$x_1 \in \delta f \text{ and } x_1 < f(x_1) \tag{1}$$

belongs to α', and if ρf is not included in δf every element x_2 such that

$$x_2 \in \delta f \text{ and } f(x_2) \notin \delta f \tag{2}$$

belongs to α'. Omitting an element x_1 from δf which satisfies (1) or an element x_2 which satisfies (2) changes $f(x)$, but not the class of all sets retraced by

$f(x)$. We call a function $f(x)$ *special* if $f(x)$ is downward and $\rho f \subset \delta f$. If the function $r(x)$ retraces α, the function $r_1(x)$ defined by

$$r_1(x) = \begin{cases} r(x) \text{ if } x \in \delta r \text{ and } r(x) < x, \\ x \text{ if } x \in \delta r \text{ and } r(x) > x, \end{cases}$$

is a downward function which retraces α. The subset δ^* of all elements $x \in \delta r_1$ such that $r_1(x), r_1r_1(x), \ldots$ are all defined (after finitely many elements all elements in this sequence are then the same) is r.e. If $r_2(x)$ is the restriction of $r_1(x)$ to δ^*, $r_2(x)$ is a special retracing function of α. Thus, *a set is retraceable if and only if it has a special retracing function.*

For every special function $f(x)$ we define

$$\text{(3)} \qquad \begin{cases} f^0(x) = x, f^{s+1}(x) = ff^s(x), \\ n(x) = (\mu s)[f^{s+1}(x) = f^s(x)], \\ \rho_{g(x)} = (f^0(x), \ldots, f^{n(x)}(x)) - (f^0(x)). \end{cases}$$

The functions $n(x)$ and $g(x)$ have the same domain as $f(x)$, and if $f(x)$ is partial recursive, so are $n(x)$ and $g(x)$. The function $g(x)$ is called the *associated function* of $f(x)$ and is denoted by $f^*(x)$.

PROPOSITION P1. *The set α is retraceable if and only if there is a partial recursive function $t(x)$ such that*

$$\text{(a)} \qquad x \in \alpha \Rightarrow \begin{cases} t(x) \text{ is defined,} \\ \rho_{t(x)} = \{y \mid y \in \alpha \text{ and } y < x\}. \end{cases}$$

Proof. We ignore the trivial case $\alpha = \phi$.

(1) Let the function $t(x)$ satisfy (a). Put

$$r(x) = \begin{cases} \min \alpha, \text{ if } \rho_{t(x)} = \phi, \text{ that is, } t(x) = 0, \\ \max \rho_{t(x)}, \text{ if } \rho_{t(x)} \neq \phi, \text{ that is, } t(x) > 0, \end{cases}$$

then $r(x)$ is a retracing function of α with the same domain as $t(x)$.

(2) Let α be retraceable. Then α has a special retracing function, say $r(x)$. If $t(x) = r^*(x)$, $t(x)$ is related to α by (a).

In the special case that α has a r.e. complement condition (a) in P1 can be replaced by

$$\text{(b)} \qquad \begin{cases} \text{there is a partial recursive function } t_1(x) \text{ such that} \\ x \in \alpha \Rightarrow \begin{cases} t_1(x) \text{ is defined,} \\ t_1(x) = \text{card } \{y \mid y \in \alpha \text{ and } y < x\}. \end{cases} \end{cases}$$

Since (a) obviously implies (b) it suffices to prove the converse. Assume (b). Let $a'(n)$ be a recursive function ranging over α' and put $u(x) = x - t_1(x)$. Let δ^* be the set of all elements x of δt_1 such that α' contains at least $u(x)$ elements $< x$. Then δ^* is a r.e. subset of δt_1 and for every $x \in \delta^*$ we can effectively find the $u(x)$ elements $b_{x1}, \ldots, b_{x,u(x)}$ which are the first $u(x)$ elements $< x$ which show up in the sequence $a'(0), a'(1), \ldots$. Thus there is a partial recursive function $q(x)$ defined on δ^* such that

$$\rho_{q(x)} = (0, \ldots, x-1) - (b_{x1}, \ldots, b_{x,u(x)}).$$

Now assume $x \in \alpha$. Among the x elements $0, \ldots, x - 1$ exactly $u(x)$ belong to α' and exactly $t_1(x)$ to α. Hence,

$$x \in \alpha \Rightarrow x \in \delta^* \text{ and } \rho_{q(x)} = \{y \mid y \in \alpha \text{ and } y < x\}.$$

We call the special function $f(x)$ *non-trivial* if the function

(4) $$m(x) = \text{card } \rho_{f^*(x)}$$

is not bounded. Every non-trivial special function has therefore an infinite domain (the converse is false as is illustrated by the identity function) and every special retracing function is non-trivial. The function described in Example 6 is non-trivial and special, but not a retracing function. Let

(5) $$\tau_n = \{x \mid m(x) = n\}.$$

The formulas (3), (4), and (5) associate with every special function $f(x)$ a sequence $\{\tau_n\}$ of sets; this sequence will be called the *sequence associated with* $f(x)$. The sets $\tau_0, \tau_1, \ldots$ are mutually disjoint for any special function $f(x)$; if $f(x)$ is non-trivial they are non-empty. In the special case that $f(x)$ is also partial recursive, the associated sequence $\{\tau_n\}$ is a r.e. sequence of mutually disjoint non-empty r.e. sets; in that case there is a recursive function $t(n, x)$ such that $\tau_n = \rho t(n, x)$ for every n. Finally, if $f(x)$ is a special retracing function, every infinite set retraced by $f(x)$ contains exactly one element of each of the sets $\tau_0, \tau_1, \ldots$.

Proposition P2. *Every recursive set is retraceable and every retraceable set is introreducible.*

Proof. The first part is obvious. Every finite set is trivially introreducible. Let α be an infinite retraceable set, $t(x)$ the function associated with a special retracing function of α and β an infinite subset of α. We claim that α is reducible to β. Put

$$g(x) = (\mu y)[x < y \text{ and } y \in \beta],$$

then $\delta g = \epsilon$, $\rho g \subset \delta t$ and $g(x)$ is recursive in $c_\beta(x)$. Moreover,

$$x \in \alpha \Leftrightarrow x \in \{y \mid y \in \alpha \text{ and } y < g(x)\} = \rho_{tg(x)};$$

hence, $c_\alpha(x)$ is recursive in $c_\beta(x)$.

Proposition P3. *Every introreducible set is recursive or immune.*

Proof. Let α be introreducible, but not immune. Either α is finite, hence recursive, or α has an infinite r.e. subset. In the latter case α also has an infinite recursive subset, say η. In that case α is recursive, because it is reducible to η.

Corollary. *Every retraceable set is recursive or immune.*

Proposition P4. *The family of all principal functions of infinite retraceable sets is closed under composition.*

Proof. Let γ be the range of $h_\alpha h_\beta$, where α and β are infinite retraceable sets. Since both h_α and h_β are strictly increasing, so is $h_\alpha h_\beta$; thus $h_\gamma = h_\alpha h_\beta$. Let $f(u)$ be the function defined on $\gamma - (h_\gamma(0))$ which maps $h_\gamma(x+1)$ onto $h_\gamma(x)$. To prove that γ is retraceable it suffices to show that $f(u)$ has a partial recursive extension. Let $t(x)$ be the function associated with a special retracing function of α and let $r(x)$ be a retracing function of β. Put

$$\begin{aligned} m(u) &= \text{card } \rho_{t(u)} \text{ for } u \in \delta t, \\ \sigma &= \{u \mid u \in \delta t \text{ and } t(u) > 0 \text{ and } m(u) \in \delta r\}. \end{aligned}$$

Then $m(u)$ is a partial recursive funcion and σ a r.e. set. We write $m = m(u)$, keeping in mind that m depends on u. For every element u in σ the m elements $a_u(0), \ldots, a_u(m-1)$ such that

$$\begin{gathered} \rho_{t(u)} = (a_u(0), \ldots, a_u(m-1)) \\ a_u(0) < a_u(1) < \ldots < a_u(m-1) \end{gathered}$$

can be effectively found. Let $g(u) = a_u r(m)$ for $u \in \sigma$, then $g(u)$ is a partial recursive function. In the special case $u = h_\gamma(x+1)$ we have:

$$\begin{aligned} a_u(0) &= h_\alpha(0), \ldots, a_u(m-1) = h_\alpha(m-1), \\ m &= h_\beta(x+1),\ r(m) = h_\beta(x), \\ g(u) &= a_u r(m) = h_\alpha r(m) = h_\alpha h_\beta(x) = h_\gamma(x) = f(u). \end{aligned}$$

Thus $g(u)$ is a partial recursive extension of $f(u)$.

Corollary. *Every infinite retraceable set has c infinite retraceable subsets.*

Proof. Let α and σ be infinite retraceable sets and $\alpha_\sigma = \rho h_\alpha h_\sigma$, then α_σ is an infinite retraceable subset of α. We know from the introduction that σ can be chosen in c different ways. The desired result now follows from the fact that different choices of σ yield different sets α_σ.

Proposition P5. *Every retraceable set with a r.e. complement has a recursive special retracing function.*

Proof. Let α be retraceable and α' r.e. If α is recursive, the function $p_\alpha(x)$ is partial recursive and the function $f(x)$ such that

$$f(x) = p_\alpha(x) \text{ for } x \in \alpha;\ f(x) = x \text{ for } x \notin \alpha,$$

is a recursive special retracing function of α. Now, assume α is not recursive; in this case both α and α' are infinite. Let $r(x)$ be a special retracing function of α and let $a'(x)$ and $d(x)$ be one-to-one recursive functions ranging over α' and δr respectively. Put $b(2n) = a'(n)$, $b(2n+1) = d(n)$, then every number occurs at least once in the sequence $b(0), b(1), \ldots$. Let us call the number x an a'-number, if $(\mu n)[x = b(n)]$ is even, otherwise a d-number. Observe that every element of α is a d-number, while α' contains both d-numbers and a'-numbers. Let

$$f(x) = \begin{cases} r(x) & \text{if } x \text{ is a } d\text{-number,} \\ x & \text{if } x \text{ is an } a'\text{-number.} \end{cases}$$

Since we can effectively determine for every number whether it is a d-number or an a'-number, we see that $f(x)$ is a recursive retracing function of α. The function $f(x)$ is downward, because $r(x)$ is downward; $\rho f \subset \delta f$ since $\delta f = \epsilon$. Thus $f(x)$ is a special function.

PROPOSITION P6. *Two disjoint retraceable sets have a common retracing function if and only if they are separable.*

Proof. Let α_1 and α_2 be disjoint and retraceable, $r_1(x)$ and $r_2(x)$ retracing functions of α_1 and α_2 respectively. We may assume without loss of generality that α_1 and α_2 are non-empty; put $m_1 = \min \alpha_1$, $m_2 = \min \alpha_2$. Assume $\alpha_1 | \alpha_2$, say $\alpha_1 \subset \beta_1$, $\alpha_2 \subset \beta_2$, where β_1 and β_2 are disjoint and r.e. Define

$$r_3(x) = \begin{cases} r_1(x) & \text{for } x \in \beta_1 \cdot \delta r_1, \\ r_2(x) & \text{for } x \in \beta_2 \cdot \delta r_2, \end{cases}$$

then $r_3(x)$ retraces both α and β. To prove the converse, assume α_1 and α_2 have a common retracing function. It is readily seen that this implies that α_1 and α_2 have a common special retracing function say $r(x)$. Let $t(x) = r^*(x)$. Put

$$\beta_1 = \{x | x \in \delta t \text{ and } \min \rho_{t(x)} = m_1\} + (m_1),$$
$$\beta_2 = \{x | x \in \delta t \text{ and } \min \rho_{t(x)} = m_2\} + (m_2),$$

then α_1 and α_2 are separated by the disjoint r.e. sets β_1 and β_2.

5. Theorems.

THEOREM T1. *There are exactly c retraceable sets; among these $\aleph_0$ are recursive, c hyperimmune and c immune, but not hyperimmune.*

Proof. Let us call a sequence $\{a_n\}$ of numbers *decimal* if $1 \leqslant a_0 \leqslant 9$ and $0 \leqslant a_n \leqslant 9$ for $n \geqslant 1$. In the introduction we defined a one-to-one correspondence between the family of all decimal sequences and a certain class of c infinite retraceable sets; we denote this correspondence by Φ. If $\sigma = \Phi < a_n >$,

$$h_\sigma(n) = 10^n \cdot a_0 + 10^{n-1} \cdot a_1 + \ldots + a_n.$$

This implies that $\Phi < a_n >$ is a recursive set if and only if a_n is a recursive function of n. There are c decimal sequences $\{a_n\}$ in which a_n is not a recursive function of n; the images of these sequences under Φ are therefore retraceable sets which are immune. None of these c sets is, however, hyperimmune, since each contains exactly one element of $(0, \ldots, 9)$, exactly one element of $(10, \ldots, 99)$ etc. We proceed to prove that there exist c retraceable sets which are hyperimmune. It suffices to prove the existence of a single retraceable set which is hyperimmune. For if θ is such a set, θ has c infinite retraceable

subsets by the Corollary of P4, and every infinite subset of a hyperimmune set is again hyperimmune. With every sequence $\{a_n\}$ of numbers we associate a set

$$\alpha = \Psi < a_n > = (a_0, a_0a_1, a_0a_1a_2, \ldots).$$

Let $\mathfrak{F}$ denote the family of all strictly increasing sequences of primes. If $\alpha = \Psi < a_n >$, where $\{a_n\} \in \mathfrak{F}$, we can for every element $x \in \alpha - (a_0)$ obtain $p_\alpha(x)$ from x by dividing x by its greatest prime factor. Thus Ψ maps $\mathfrak{F}$ onto a class of infinite retraceable sets. Let $p_0, p_1, \ldots$ be the sequence of all primes arranged according to size. Suppose $t(n)$ is the principal function of a hyperimmune set. Put $q_n = p_{t(n)}$, $\theta = \Psi < q_n >$, $r_n =$ the principal function of θ, then $\{q_n\} \in \mathfrak{F}$ and θ is an infinite retraceable set. The function $t(n)$ is not bounded by any recursive function, because it is the principal function of a hyperimmune set (7, Theorem 21). Taking into account that $t(n) < q_n \leqslant r_n$, it follows that r_n is not bounded by any recursive function. This implies that θ is hyperimmune.

In the following we write $\alpha \equiv_T \beta$ if α and β are (Turing) reducible to each other, that is, if $c_\alpha(x)$ and $c_\beta(x)$ are recursive in each other. We shall use the well-known functions $j(x, y)$, $k(z)$, $l(z)$ defined by

$$\begin{aligned} j(x, y) &= x + \tfrac{1}{2}(x + y)(x + y + 1), \\ k(z) &= (\mu x)(\exists y)[j(x, y) = z], \\ l(z) &= (\mu y)(\exists x)[j(x, y) = z]. \end{aligned}$$

THEOREM T2. *Every degree of unsolvability can be represented by a retraceable set.*

Proof. Let α be any set. We wish to associate with α a retraceable set β such that $\beta \equiv_T \alpha$. If α is finite we can take $\beta = \alpha$; if $0 \in \alpha$ we can replace α by $\alpha - (0)$, since $\alpha \equiv_T \alpha - (0)$. We may therefore assume without loss of generality that α is infinite and does not contain 0. Let $a_n = h_\alpha(n)$, then $a_0 > 0$, and hence $a_n > n$. Put

(6) $$b_0 = a_0, \quad b_{n+1} = j(b_n, a_{n+1}), \quad \beta = \rho b_n.$$

Before proving that β satisfies the requirements we first observe that

(7) $$0 < b_0 < b_1 < \ldots; \quad b_n > n.$$

For it follows from the definition of $j(x, y)$ that $x < j(x, y)$ for $y \neq 0$. Thus, since $a_{n+1} > n + 1 > 0$, we have $b_n < j(b_n, a_{n+1}) = b_{n+1}$; moreover, $b_0 = a_0 > 0$. Hence the first part of (7) holds and this implies $b_n > n$. The function

$$f(x) = \begin{cases} b_0 & \text{for } x = b_0 \\ k(x) & \text{for } x \neq b_0 \end{cases}$$

is a recursive extension of $p_\beta(x)$ in view of (6). This shows that β is retraceable.

The relation $a_x > x$ implies

$$x \in \alpha \Leftrightarrow x \in (a_0, a_1, \ldots, a_x), \tag{8}$$
$$x \in \alpha \Leftrightarrow x \in (b_0, l(b_1), \ldots, l(b_x)).$$

The elements $b_0, \ldots, b_x$ can be found by computing $c_\beta(0), c_\beta(1), \ldots$ until the value 1 has shown up $x + 1$ times. Hence $c_\alpha(x)$ is recursive in $c_\beta(x)$ by (8). From $b_y > y$ we infer

$$y \in \beta \Leftrightarrow y \in (b_0, b_1, \ldots, b_y). \tag{9}$$

In view of (6) the elements $b_0, b_1, \ldots, b_y$ can be effectively computed from $a_0, a_1, \ldots, a_y$; but $a_0, a_1, \ldots, a_y$ can be determined by computing $c_\alpha(0), c_\alpha(1), \ldots$ until the value 1 has shown up $y + 1$ times. Thus $c_\beta(x)$ is recursive in $c_\alpha(x)$ by (9) and β is a retraceable set such that $\beta \equiv_T \alpha$.

Corollary. *Every degree of unsolvability higher than the lowest degree (that is, than the degree consisting of all recursive sets) can be represented by an immune set.*

Remark. The proposition that there exist exactly c retraceable steps can also be obtained as a corollary of T2. For every degree of unsolvability consists of $\aleph_0$ sets and the total number of sets is c. Thus there are c degrees of unsolvability, hence, at least c (and therefore exactly c) retraceable sets.

Among the degrees of unsolvability those which can be represented by r.e. sets are of special interest. These degrees can, of course, also be defined as those which can be represented by sets which a r.e. complement, because $\alpha \equiv_T \alpha'$ for every set α. We shall see in T3 that every degree of this type can be represented by a retraceable set with a r.e. complement. For every function $f(x)$ we define

$$\zeta_f = \{x | (\exists y)[x < y \text{ and } f(x) > f(y)]\}.$$

The set σ is a *deficiency set*, if $\sigma = \zeta_f$ for some one-to-one recursive function $f(x)$; σ is a *deficiency set of the infinite r.e. set* α if $\sigma = \zeta_a$ for some one-to-one recursive function $a(x)$ ranging over α.

Theorem T3. *Every degree of unsolvability which can be represented by a r.e. set can also be represented by a r.e. set with a retraceable complement.*

Proof. Let α be a r.e. set. We wish to associate with α a r.e. set β such that $\beta \equiv_T \alpha$ and β' is retraceable. If α is recursive, α' is also recursive and therefore retraceable. We may therefore assume that α is not recursive. Let a_n be a one-to-one recursive function ranging over α and let $\beta = \zeta_a$. Then β is a r.e. set such that $\beta \equiv_T \alpha$ by (**2**, Theorem 1). We claim that β' is retraceable. Let us denote the finite sequence $\{a_0, \ldots, a_m\}$ by $\Sigma(m)$. Our proof is based on

$$x \in \beta' \Rightarrow \left[z < x \text{ and } z \in \beta' \Leftrightarrow \begin{cases} \text{(a)} & a_z \text{ occurs in } \Sigma(x-1), \\ (b) & a_z < \text{each of its successors in } \Sigma(x) \end{cases} \right] \tag{10}$$

To establish (10) assume $x \in \beta'$, that is, $(\forall y)[x < y \Rightarrow a_x < a_y]$. If $x = 0$ we interpret $\Sigma(x-1)$ as the empty sequence. In that case (a) is false for every z and so is $z < x$ and $z \in \beta'$. We may therefore assume $x > 0$. Suppose $z < x$ and $z \in \beta'$. Then $z \leq x-1$ and a_z occurs in $\Sigma(x-1)$. Moreover, since $z \in \beta'$, a_z is less than each of its successors in $\{a_n\}$, in particular less than each of its successors in $\Sigma(x)$. Thus z satisfies both (a) and (b). Conversely, assume that z satisfies (a) and (b). Then $z < x$ because z satisfies (a), and

(11) $$a_z < a_{z+1}, a_z < a_{z+2}, \ldots, a_z < a_x,$$

because z satisfies (b). Also

(12) $$a_z < a_{x+i} \text{ for } i \geq 1,$$

in view of $x \in \beta'$. Combining (11) and (12) we see that $z \in \beta'$. This completes the proof of (10). Whether $x \in \beta$ or $x \notin \beta$, the set

$$\{z | a_z \text{ satisfies (a) and (b)}\}$$

can be effectively obtained from x. Thus there is a recursive function $t(x)$ such that

$$\rho_{t(x)} = \{z | a_z \text{ satisfies (a) and (b)}\}.$$

By (10)

$$x \in \beta' \Rightarrow \rho_{t(x)} = \{z | z < x \text{ and } z \in \beta'\}.$$

We conclude by P1 that β' is retraceable.

Let α be any r.e., but not recursive set and let β be one of the deficiency sets of α. We have seen in the proof of T3 that β' is retraceable and it was shown in **(2)** that β is hypersimple. The set β' is therefore an example of a hyperimmune retraceable set with a r.e. complement. By P1 there exist retraceable sets which are immune, but not hyperimmune. The question arises whether such sets can have a r.e. complement. The answer is negative according to the following theorem.

THEOREM T4. *Every retraceable set with a r.e. complement is recursive or hyperimmune.*

Proof. Let α be retraceable and α' r.e. If α is finite, it is recursive. We may therefore assume that α is infinite. All we have to show is:

(13) $$\alpha \text{ not hyperimmune} \Rightarrow \alpha \text{ recursive}.$$

The conclusion of (13) is equivalent to the assertion that α is not immune, in view of the retraceability of α. Since α is infinite, α is not immune if and only if α has an infinite r.e. subset. Moreover, α has an infinite r.e. subset if and only if there is an effective procedure which, given any number k, enables us to find k distinct elements of α. Instead of proving (13) we can therefore prove:

(14) *If α is not hyperimmune there is an effective procedure such that given any number k we can find k distinct elements of α.*

Assume α is not hyperimmune. Using the fact that α is infinite we infer that there is a discrete array each of whose rows contains at least one element of α, say $\{\delta_n\}$. Let k be any number. Put $n_k = \max(\delta_0 + \ldots + \delta_k)$, then $\delta_0 + \ldots + \delta_k$ contains at least $k + 1$ elements of α which are $\leqslant n_k$. Each of the elements $0, 1, \ldots, n_k - 1$ occurs in at most one row of $\{\delta_n\}$; thus, at most finitely many rows of $\{\delta_n\}$ contain an element $< n_k$ and we can effectively find the first row of $\{\delta_n\}$ all of whose elements are $\geqslant n_k$, say

$$\delta = (d(0), \ldots, d(s)), \text{ where } d(0) < d(1) < \ldots < d(s).$$

Note that the elements of δ and s depend on k. Row δ contains at least one element and at most $s + 1$ elements of α. Let $d(w) = \min \alpha \cdot \delta$. The set α has a recursive special retracing function, since α' is r.e., say $r(x)$; the function $t(x) = r^*(x)$ is therefore also recursive. Put

$$\sigma_0 = \rho_{td(0)} \cdot \rho_{td(1)} \cdot \ldots \cdot \rho_{td(s)}.$$

We now infer from

$$d(w) \in \alpha \Rightarrow \rho_{td(w)} = \{y | y \in \alpha \text{ and } y < d(w)\}$$
$$d(w) \in \delta \Rightarrow d(w) \geqslant n_k \Rightarrow \text{card } \rho_{td(w)} \geqslant k$$

that

$$\sigma_0 \subset \rho_{td(w)} \subset \alpha \text{ and card } \rho_{td(w)} \geqslant k.$$

If card $\sigma_0 \geqslant k$ we are through, because $\sigma_0 \subset \alpha$. If card $\sigma_0 < k$ at least one of the sets $\rho_{td(0)}, \rho_{td(1)}, \ldots, \rho_{td(s)}$ does not contain the k smallest elements of α; since $d(0), \ldots, d(s)$ are all $\geqslant n_k$ this means that at least one of these $s + 1$ elements does not belong to α. Let $a'(n)$ be a recursive function generating α'; by computing $a'(0), a'(1), \ldots$ we can effectively find the first element of $\{a'(n)\}$ which belongs to δ, say $d(p)$. Put

$$\sigma_1 = \prod \rho_{td(i)},$$

i ranging over $(0, \ldots, s) - (p)$, then $\sigma_1 \subset \rho_{td(w)} \subset \alpha$ and card $\rho_{td(w)} \geqslant k$. Again, if card $\sigma_1 \geqslant k$ we are through; if card $\sigma_1 < k$ we generate $\alpha' = (a'(0), a'(1), \ldots)$ until an element of $\alpha' \cdot \delta - (d(p))$ is obtained, say $d(q)$; we then define

$$\sigma_2 = \prod \rho_{td(i)},$$

i ranging over $(0, \ldots, s) - (p, q)$, etc. The set δ contains only $s + 1$ elements, hence the procedure must terminate. This means that after a finite number of steps we obtain a set σ_u such that $\sigma_u \subset \rho_{td(w)} \subset \alpha$ and card $\sigma_u \geqslant k$. Then we have found k elements of α, though in general, we don't know whether $\sigma_u = \rho_{td(w)}$. Thus (14) is proved.

COROLLARY. *There exist immune sets which are not retraceable, but have a r.e. complement.*

Proof. If ζ is simple, but not hypersimple, ζ' satisfies the requirements.

Remark. Let us call a simple set σ *extendible* if there is a simple set τ such that $\sigma \subset \tau$ and $\tau - \sigma$ is infinite. A simple, but not hypersimple, set is always extendible. For suppose $\{\delta_n\}$ is a discrete array each of whose rows contains at least one element of the complement of the simple set σ. Put $\tau = \sigma + \Sigma_0^\infty \delta_{2n}$, then τ is a simple set which includes σ and is such that $\tau - \sigma$ is infinite. The question was raised (**9**, p. 215, Problem 9), whether there exists a simple set which is not extendible. We have just shown that if such a simple set exists, it must be hypersimple. We now prove that the hypersimple sets discussed above, namely, those with a retraceable complement, are not candidates for non-extendibility. For, assume ζ is hypersimple and ζ' is retraceable. Let $r(x)$ be a recursive special retracing function of ζ' and let $\{\tau_n\}$ be the sequence of sets associated with $r(x)$. Put $\zeta^* = \zeta + \Sigma_0^\infty \tau_{2n}$, then ζ^* is a hypersimple set such that $\zeta \subset \zeta^*$ and $\zeta^* - \zeta$ is infinite.

THEOREM T5. *There exist two retraceable immune sets α and β such that* (1) $\alpha|\beta$, (2) $\alpha + \beta$ *is not retraceable, and* (3) $\alpha + \beta$ *is introreducible.*

Proof. Let $\{f_n\}$ be a sequence of numbers chosen from $(1, \ldots, 9)$ with $f_0 = 2$ and such that f_n is not a recursive function of n. Put

$$a_0 = f_0,\ a_{n+1} = 10 \cdot a_n + f_{n+1},\ \alpha = \rho a_n,\ b_n = 10^{a_n},\ \beta = \rho b_n.$$

We claim that α and β satisfy the requirements. The set α is clearly retraceable and immune; 10^x is a strictly increasing recursive function which maps α onto β and hence β is also retraceable and immune. Put $\eta = (10, 10^2, \ldots)$, then α and β can be separated by the recursive sets η' and η; this proves (1). We define

$$\delta_n = (10^n + 1, 10^n + 2, \ldots, 10^{n+1} - 1).$$

Note that α has exactly one element in common with each row of the discrete array $\{\delta_n\}$. From $x < 10^{x-1}$ for $x > 2$ we conclude: *For every number $x > 2$ there is exactly one number y such that*

$$x < 10^{x-1} < y < 10^x \textit{ and } y \in \alpha. \tag{15}$$

Suppose $\alpha + \beta$ were retraceable and $g(x)$ were one of its retracing functions. Put

$$y_0 = 2,\ y_1 = g(10^2),\ y_{n+1} = g(10^{y_n}).$$

We wish to prove that y_n is a strictly increasing recursive function all of whose values belong to α. First of all, $2 \in \alpha$ and $10^2 \in \beta \subset \alpha + \beta$. Thus y_1 is defined; in fact, since 10^2 is the minimum of β, y_1 is the unique element of α which lies between 10 and 10^2. Hence $y_0 < y_1$, where $y_0, y_1 \in \alpha$. Now assume $2 < y_n \in \alpha$, then $10^{y_n} \in \beta$; by (15) there is exactly one number $z \in \alpha$ such that

$$y_n < 10^{y_n - 1} < z < 10^{y_n}. \tag{16}$$

There is no number $z \in \beta$ which satisfies (16); the unique number $z \in \alpha$ which satisfies (16) is therefore

$$g(10^{y_n}).$$

Thus y_n is defined and $y_n < y_{n+1} \in \alpha$. The function y_n generates therefore an infinite recursive subset of the immune set α; this is a contradiction and hence (2) is correct. The sets σ_1 and σ_2 are *recursively equivalent* if σ_2 is the image of σ_1 under some partial recursive one-to-one function. We recall that β is the image of α under the recursive one-to-one function 10^x. To prove (3) it is therefore sufficient to prove the following lemma: *if the retraceable sets σ_1 and σ_2 are separable and recursively equivalent, their sum is introreducible.* Assume the hypothesis of the lemma. Let $r_1(x)$ and $r_2(x)$ be special retracing functions of σ_1 and σ_2 respectively, $t_1(x) = r_1^*(x)$, $t_2(x) = r_2^*(x)$. Suppose θ_1 and θ_2 are disjoint r.e. sets such that $\sigma_1 \subset \theta_1$ and $\sigma_2 \subset \theta_2$; assume finally that $p(x)$ is a partial recursive one-to-one function related to σ_1 and σ_2 by $\sigma_1 \subset \delta p$ and $\sigma_2 = p(\sigma_1)$. To show that $\sigma_1 + \sigma_2$ is introreducible, assume that γ is an infinite subset of $\sigma_1 + \sigma_2$. Then $\gamma \subset \theta_1 + \theta_2$ and $\gamma = \gamma \cdot \theta_1 + \gamma \cdot \theta_2$. Define

$$\gamma_1 = \gamma \cdot \theta_1 + p^{-1}(\gamma \cdot \theta_2), \quad \gamma_2 = \gamma \cdot \theta_2 + p(\gamma \cdot \theta_1).$$

It follows that γ_1 is an infinite subset of σ_1 and γ_2 an infinite subset of σ_2. If we could compute $c_\gamma(0), c_\gamma(1), \ldots$ we could also generate the sets $\gamma \cdot \theta_1$, $\gamma \cdot \theta_2$, $p(\gamma \cdot \theta_1)$, $p^{-1}(\gamma \cdot \theta_2)$ and hence γ_1 and γ_2. For any number x, let the smallest numbers y and z such that

$$y \in \gamma_1 \text{ and } y > x \text{ and } z \in \gamma_2 \text{ and } z > x$$

be denoted by $n_1(x)$ and $n_2(x)$ respectively. Thus $n_1(x)$ and $n_2(x)$ are everywhere defined functions recursive in $c_\gamma(x)$ such that

$$n_1(x) \in \gamma_1 \text{ and } x < n_1(x) \text{ and } n_2(x) \in \gamma_2 \text{ and } x < n_2(x).$$

This implies

$$x < n_1(x) \in \sigma_1 \quad \text{and} \quad x \in \sigma_1 \Leftrightarrow x \in \rho_{t_1 n_1(x)},$$
$$x < n_2(x) \in \sigma_2 \quad \text{and} \quad x \in \sigma_2 \Leftrightarrow x \in \rho_{t_2 n_2(x)}.$$

We conclude that the characteristic function of $\sigma_1 + \sigma_2$ is recursive in $c_\gamma(x)$.

COROLLARY 1. *The product of two retraceable sets is again retraceable, but the sum of two separable retraceable sets is not necessarily retraceable.*

COROLLARY. 2 *There exist introreducible sets which are not retraceable.*

A sufficient condition that two distinct retraceable sets have a common retracing function is that they are separable (by P6), but this condition is not necessary. For the infinite recursive sets $(0, 1, 2, 3, 5, 7, 9, \ldots)$ and $(0, 1, 2, 3, 4, 6, 8, \ldots)$ have a common retracing function and are not disjoint. Let us for any set σ and any number m denote the set $\{x \mid x \in \sigma \text{ and } x < m\}$ by $\sigma < m >$.

THEOREM T6. *Let α and β be distinct retraceable sets. Then α and β have a common retracing function if and only if either*

(a) $\alpha \cdot \beta = \phi$ and $\alpha | \beta$, *or*
(b) $\alpha \cdot \beta \neq \phi$ and $\max(\alpha \cdot \beta) = k \Rightarrow$

$$[\alpha < k > = \beta < k > \text{ and } \alpha - \alpha < k > \mid \beta - \beta < k >].$$

Proof. Let $\alpha \neq \beta$. Suppose α and β have a common special retracing function, say $r(x)$; put $t(x) = r^*(x)$. We claim that $\alpha \cdot \beta$ must be finite. For if $\alpha \cdot \beta$ is infinite there is a strictly increasing function ranging over $\alpha \cdot \beta$ say $c(n)$. Since $c(n) \in \alpha$ for every n, $\rho_{tc(n)} \subset \alpha$ for every n; moreover, for every $x \in \alpha$ there is an n_x such that $x < c(n_x) \in \alpha$. Hence $\alpha = \Sigma_0^\infty \rho_{tc(n)}$. Similarly one proves $\beta = \Sigma_0^\infty \rho_{tc(n)}$. Hence $\alpha = \beta$, contrary to the assumption $\alpha \neq \beta$. We conclude that $\alpha \cdot \beta$ is finite. Either $\alpha \cdot \beta = \phi$ or $\alpha \cdot \beta \neq \phi$. In the former case $\alpha|\beta$ by P6. In the latter case we put

$$\begin{aligned} &k = \max(\alpha \cdot \beta),\ a_1 = \alpha - \alpha <k>,\ \beta_1 = \beta - \beta <k>, \\ &a_0 = \min \alpha_1,\ b_0 = \min \beta_1, \end{aligned} \tag{17}$$

and

$$r_1(x) = \begin{cases} r(x) & \text{for } x \in \delta r - (0, \ldots, k, a_0, b_0), \\ a_0 & \text{for } x = a_0, \\ b_0 & \text{for } x = b_0. \end{cases}$$

Every set obtained from a retraceable set by omitting finitely many of its elements is again retraceable. Thus α_1 and β_1 are disjoint retraceable sets. They have the common retracing function $r_1(x)$. Hence $\alpha_1|\beta_1$ by P6. To prove the converse we assume that α and β are retraceable sets satisfying (a) or (b). If they satisfy (a) we are through. If they satisfy (b) we define $k, \alpha_1, \beta_1, a_0$, and b_0 as in (17). The sets α_1 and β_1 are retraceable, because α and β are retraceable; since $\alpha_1|\beta_1$ they have a common retracing function, say $r(x)$. Let $\alpha <k> = (c_0, \ldots, c_p)$, where $c_0 < c_1 < \ldots < c_p$. Put

$$r_0(x) = \begin{cases} r(x) & \text{for } x \in \delta r - (0, \ldots, k, a_0, b_0), \\ c_p & \text{for } x \in (a_0, b_0), \\ c_n & \text{for } x = c_{n+1} \text{ and } 0 \leqslant n \leqslant p - 1, \\ c_0 & \text{for } x = c_0. \end{cases}$$

Then $r_0(x)$ is a common retracing function of α and β.

Let $r(x)$ be any retracing function and T_r the class of all infinite sets retraced by $r(x)$. We know from the Examples 1–5 that given any of the cardinalities $1, 2, 3, \ldots, \aleph_0, c$, the retracing function $r(x)$ can be chosen in such a manner that T_r has the given cardinality. Assuming the continuum hypothesis, these are obviously the only values which card T_r can assume.

THEOREM T7. *It can be proved without the continuum hypothesis that the class of all infinite sets retraced by a retracing function $r(x)$ is either finite or denumerable, or has cardinality c.*

Proof. All references in this proof are to Sierpinski's book **(8)**. We use the numbers 0, 1, . . . as indices of the elements of a sequence, while Sierpinski uses 1, 2, . . . for the same purpose; this difference in notation is, however, non-essential for the theorems in Sierpinski's book which we shall use. Throughout this proof the agreement that only collections of non-negative integers are referred to as sets is suspended; any collection of points in a metric space is called a set. Let $\mathfrak{E}_\omega$ be the space consisting of all sequences of real numbers, where for $p = \{p_x\}$ and $q = \{q_x\}$,

$$\rho(p, q) = \sum_{x=0}^{\infty} \frac{|p_x - q_x|}{x!(1 + |p_x - q_x|)}.$$

Also let $\mathfrak{N}_\omega$ be the space consisting of all sequences of non-negative integers with the same distance function as $\mathfrak{E}_\omega$. $\mathfrak{E}_\omega$ is a metric space **(8**, p. 134**)** and $\mathfrak{N}_\omega$ is a subspace of $\mathfrak{E}_\omega$. We need three lemmas.

LEMMA 1. *The sequence* $\{p^n\} = \{\{p_x^n\}\}$ *of points in* $\mathfrak{N}_\omega$ *converges to the point* p *of* $\mathfrak{E}_\omega$ *if and only if*

$$(\forall x)(\exists t)(\forall n)[n > t \Rightarrow p_x^n = p_x].$$

Proof. The sequence $\{p^n\}$ of points in $\mathfrak{E}_\omega$ converges to the point p of $\mathfrak{E}_\omega$ if and only if for every x,

$$\lim_{n\to\infty} p_x^n = p_x$$

(8, p. 135**)**. The desired result follows from the fact that $p^n \in \mathfrak{N}_\omega$ means that p_x^n is a non-negative integer for every x.

LEMMA 2. *It can be proved without the continuum hypothesis that every closed set in* $\mathfrak{N}_\omega$ *is finite, denumerable or has cardinality* c.

Proof. Let $\mathfrak{N}_0$ denote the set of all points $\{q_x\}$ in $\mathfrak{N}_\omega$ which are ultimately vanishing sequences (that is, for which $q_x = 0$ for almost all x). Let $p = \{p_x\} \in \mathfrak{N}_\omega$. Put

$$q_x^n = \begin{cases} p_x \text{ for } x < n, \\ 0 \text{ for } x > n, \end{cases} \qquad q^n = \{q_x^n\},$$

then $\{q^n\}$ is a sequence of points in $\mathfrak{N}_0$ which converges to p. Thus $\mathfrak{N}_\omega$ is the closure of its denumerable subset $\mathfrak{N}_0$; hence $\mathfrak{N}_\omega$ is separable. It follows from Lemma 1 that if a sequence of points in $\mathfrak{N}_\omega$ converges in $\mathfrak{E}_\omega$, its limit belongs to $\mathfrak{N}_\omega$. Thus $\mathfrak{N}_\omega$ is a closed set in the metric space $\mathfrak{E}_\omega$. However, $\mathfrak{E}_\omega$ is complete **(8**, pp. 190, 191**)**, hence $\mathfrak{N}_\omega$ is also complete. Let $\mathfrak{B}$ be a closed set in $\mathfrak{N}_\omega$, then $\mathfrak{B}$ is a Borel set in a separable complete space, hence

$$\text{card } \mathfrak{B} > \aleph_0 \Rightarrow \text{card } \mathfrak{B} = c \tag{18}$$

by **(8**, p. 228, Corollary 2, Theorem 120**)**. This means that $\mathfrak{B}$ is finite, denumerable or has cardinality c. Moreover, (18) can be proved without using the continuum hypothesis.

LEMMA 3. *Let Δ be a denumerable collection, let $\mathfrak{A}$ be a family of finite sequences of elements of Δ and let $\mathfrak{B}$ be a family of infinite sequences of elements of Δ. Assume, furthermore, that $g = \{g_x\}$ belongs to $\mathfrak{B}$ if and only if all its initial segments $\{g_x\}_{x<n}$ belong to $\mathfrak{A}$. Then it can be proved without the continuum hypothesis that $\mathfrak{B}$ is finite, denumerable or has cardinality c.*

Proof. We may clearly restrict ourselves to the special case $\Delta = \epsilon$; in this case $\mathfrak{B}$ is a set in $\mathfrak{N}_\omega$ and by Lemma 2 it suffices to prove that $\mathfrak{B}$ is closed. Let the point $g = \{g_x\}$ in $\mathfrak{N}_\omega$ be a limit point of $\mathfrak{B}$ and let $\{g_x^n\}$ be a sequence of points in $\mathfrak{B}$ which converges to g. By Lemma 1 there is a function $t(x)$ such that

$$n > t(x) \Rightarrow g_x^n = g_x. \tag{19}$$

If g were not in $\mathfrak{B}$, it would have an initial segment not belonging to $\mathfrak{A}$, say $\{g_x\}_{x<m}$. Thus

$$\{g_x\}_{x<m} \notin \mathfrak{A}. \tag{20}$$

Let t be the number which exceeds the maximum of $t(0), \ldots, t(m)$ by 1, then $t > t(x)$ for $x < m$. Hence, by (19), $g_x^t = g_x$ for $x < m$, that is,

$$\{g_x^t\}_{x<m} = \{g_x\}_{x<m}. \tag{21}$$

We now have a contradiction. For, since the left side of (21) is an initial segment of the point g^t in $\mathfrak{B}$, it belongs to $\mathfrak{A}$; on the other hand, the right side of (21) does not belong to $\mathfrak{A}$ by (20). We conclude that $g \in \mathfrak{B}$. Hence $\mathfrak{B}$ is a closed set in $\mathfrak{N}_\omega$.

We claim that T7 follows from Lemma 3. Assume that $f(x)$ is any function with an infinite domain and B the class of all infinite sets β of non-negative integers such that $f(x)$ is an extension of $p_\beta(x)$. Since T7 concerns the special case that $f(x)$ is partial recursive and B non-empty, it suffices to prove that B is finite, denumerable or has cardinality c. Let $\mathfrak{B}$ denote the family of the principal functions of all sets in B and $\mathfrak{A}$ the family of all strictly increasing finite sequences $\{x_0, \ldots, x_n\}$ such that

$$\alpha = (x_0, \ldots, x_n) \Rightarrow f(x) \text{ is an extension of } p_\alpha(x).$$

Clearly, $g = \{g_x\} \in \mathfrak{B}$ if and only if all its initial segments belong to $\mathfrak{A}$. Thus it follows by Lemma 3 that the family $\mathfrak{B}$ is finite, denumerable, or has cardinality c; the same is therefore true for the class B.

Remark. Lemma 3 can be used to establish the following theorem about graphs. *Let Γ be a graph with denumerably many edges; then it can be proved without the continuum hypothesis that the number of one-way infinite paths of Γ is finite, $\aleph_0$ or c.* For let Δ denote the collection of all edges of Γ, $\mathfrak{A}$ the family of all finite paths of Γ and $\mathfrak{B}$ the family of all one-way infinite paths of Γ. Then Lemma 3 is applicable, since the infinite sequence $\{p_0p_1, p_1p_2, \ldots\}$ of edges belongs to $\mathfrak{B}$ if and only if all its initial segments $\{p_0p_1, \ldots, p_np_{n+1}\}$ belong to $\mathfrak{A}$.

6. Concluding remarks. We have not been able to characterize all partial recursive functions which are retracing functions, that is, which retrace at least one infinite set. Let us denote by T_f the class of all infinite sets retraced by $f(x)$. For every retracing function $f(x)$ there is a non-trivial special partial recursive function $r(x)$ which is a restriction of $f(x)$ and which has the property $T_r = T_f$. We can therefore restate the problem as follows: find a necessary and sufficient condition that a non-trivial special partial recursive function be a retracing function. We mention a sufficient condition which is not necessary. *If a non-trivial, special, partial recursive function has a finite initial set and is finite-to-one, it is a retracing function.* For assume $f(x)$ satisfies the hypotheses. Let $\{\tau_n\}$ be the infinite sequence of (mutually disjoint non-empty) sets associated with $f(x)$, then $\{\tau_n\}$ is a sequence of finite sets. Define a binary relation R by: yRx if $y = f(x)$. Since there corresponds with every element x of τ_{n+1} at least one (in fact, exactly one) element y of τ_n such that yRx, it follows by König's Lemma (**4**, p. 121 or **5**, p. 81) that there exists an infinite sequence $\{a_n\}$ such that for every n, a_nRa_{n+1}. Then $f(x)$ retraces the infinite set $(a_0, a_1, \ldots)$ and the proof is completed. This immediately raises the question: "Does every function $f(x)$ which satisfies this sufficient condition retrace at least one infinite *recursive* set?" One might be tempted to conjecture that the answer is affirmative, because Brouwer's proof (**3**, p. 42) of his fan theorem, that is, the intuitionistic form of König's lemma, is in some sense constructive. Though it can hardly be doubted that a close connection exists between König's lemma and the subject of the present paper, the authors have, however, been unable to substantiate this conjecture, even in the special case that given any $x \in \rho f$ the cardinality of the set $f^{-1}(x)$ can effectively be found. In this case $\{\tau_n\}$ is a discrete array.

Added May 31, 1958. It is proved in a paper of R. M. Friedberg which will appear in the Journal of Symbolic Logic that there exists a simple set which is not extendible.

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ON CONJUGATES IN DIVISION RINGS

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Let D be a non-commutative division ring with centre C, and let Δ be a proper division subring not contained in C. In **(4)** Cartan raised the question: is it ever possible for each inner automorphism of D to induce an automorphism of Δ? As is well-known, Cartan **(4**, Théorème 4**)** with the aid of his Galois Theory answered this negatively in case D is a finite dimensional division algebra. Later Brauer **(3)**, and Hua **(8)**, using elegant, elementary methods, extended Cartan's theorem to arbitrary division rings.

Let D^* denote the group of all non-zero elements of D, and let $H(\Delta)$ be the subgroup of all elements of D^* which effect inner automorphisms of D that map Δ onto Δ. In this note I prove the following extension of the Cartan-Brauer-Hua theorem: *$H(\Delta)$ cannot have finite index in D^**. This theorem implies (and is implied by) the condition: *D always contains infinitely many subrings $x\Delta x^{-1}$ isomorphic* (*or conjugate*) *to Δ.*

Although this result implies that every finite division ring is commutative, its proof does not constitute a new proof of this old theorem **(17)** of Wedderburn's. As a matter of fact, the proof requires not only Wedderburn's theorem but also Jacobson's theorem **(9)** on algebraic division algebras over a finite field.

1. Conjugates in division rings. If S is any subset of a division ring D, the *centralizer of S in D* is the set $S' = \{x \in D | sx = xs \text{ for all } s \in S\}$. When S consists of the single element θ, θ' denotes this division subring. S'' is the division subring $(S')'$. If Δ is any division subring of D, Δ^* represents the multiplicative group of non-zero elements of Δ. C will always be the centre of D.

The group of all automorphisms of D which leave fixed each element of Δ is signified by $G(\Delta)$; $J(\Delta)$ is the subgroup of those inner automorphisms of D which belong to $G(\Delta)$. The group $G(\Delta)$ is *outer* when $J(\Delta)$ is the identity subgroup (e). Since $J(\Delta)$ is isomorphic to ∇^*/C^*, where $\nabla = \Delta'$, one deduces from the following proposition that if $J(\Delta)$ is a finite group $\neq (e)$, then Δ' is a finite field. Thus when C is not a finite field, $J(\Delta)$ is finite if and only if $G(\Delta)$ is outer.

PROPOSITION 1. *If ∇ is any proper division subring of a division ring D, then ∇^* has finite index in D^* if and only if D is a finite field.*

Proof. If D is a finite field there is nothing to prove. Conversely, if D is not a finite field, then D is not finite **(17)**. Suppose ∇^* has finite index in D^*. Then

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∇^* necessarily has infinitely many elements; for each $\theta \in D$, there exist elements δ_1, δ_2, and $\delta \in \nabla$, $\delta_1 \neq \delta_2$, such that $\theta + \delta_1 = \delta(\theta + \delta_2)$. But then $(1 - \delta)\theta \in \nabla$. Since $\delta_1 \neq \delta_2$, δ cannot be 1. Thus $(\delta - 1)^{-1} \in \nabla$, so that $\theta \in \nabla$. Hence $D = \nabla$, and ∇ is not a proper subring.

Proposition 1 actually implies that a non-central element θ of a non-commutative division ring D has infinitely many conjugates in D. This is Herstein's theorem **(7)**. As several authors **(15, 16)** have remarked, the Cartan-Brauer-Hua theorem is not needed in the proof.

If Δ is an arbitrary division subring of D, there is occasion to consider isomorphisms of $\Delta(\theta)$ which leave fixed the elements of Δ. Such an isomorphism, often called *an isomorphism of* $\Delta(\theta)$ *with respect to* Δ, when induced by an inner automorphism of D is effected by an element $x \in \Delta'$. If θ commutes with every element of Δ', that is, if $\theta \in \Delta''$, then no non-trivial isomorphism of the kind mentioned exists. If Δ' is finite, then the number of conjugates $x^{-1}\theta x$, with $x \in \Delta'$, is also finite. In all other cases, however, θ has infinitely many such conjugates, as can be deduced from the case $\nabla = \Delta'$ of the next theorem, which has been obtained also by Kasch **(11)**.

THEOREM 1. *Let D be a non-commutative division ring, and let ∇ be an infinite division subring not contained in the center of D. Then every element θ in D which is outside of ∇' possesses infinitely many conjugates $x^{-1}\theta x$ with $x \in \nabla$.*

Proof. If θ has only finitely many conjugates with $x \in \nabla$, then A^* has finite index in ∇^*, where $A = \nabla \cap \theta'$. Since ∇ is not finite, by Proposition 1, A must be all of ∇. But then $\nabla' = A'$. Since $\theta \in A'$ this implies that $\theta \in \nabla'$, contrary to its choice. Thus θ must have infinitely many conjugates $x^{-1}\theta x$, with $x \in \nabla$.

The following corollaries are all proved under the assumption of Theorem 1, that is, $\nabla = \Delta'$ is infinite. In case D is an algebraic division algebra, Jacobson's theorem **(9)** makes this assumption on Δ' superfluous.

COROLLARY. *Let $[\theta]$ denote the set of elements in D of the form $x\theta x^{-1}$, with $x \in \Delta'$. Then, if $[\theta]$ contains an element other than θ, then $[\theta]$ contains infinitely many elements.*

Proof. Since $[\theta] \neq \theta$, then $\theta \notin \Delta''$, so by Theorem 1, the set $[\theta]$ is infinite.

If $[\theta]$ is finite for every $\theta \in D$, then $D = \Delta''$. Thus $\Delta' = C$, that is, $G(\Delta)$ is outer. This yields the next corollary, which emphasizes the severity of Nobusawa's locally finite condition **(13)**.

COROLLARY. *If the set of conjugates of θ with respect to Δ is finite for each $\theta \in D$, then $G(\Delta)$ is outer.*

Let X be indeterminate over Δ, and let $\Delta[X]$ denote the polynomial ring consisting of all finite sums of elements of the form $aXbX \ldots cXd$, with $a, b, \ldots, c, d, \in \Delta$. If $\theta \in D$ is a zero of a polynomial in $\Delta[X]$, then so is

every conjugate of θ with respect to Δ. Thus the next corollary is a consequence of Theorem 1. When $\Delta = C$ it specializes to a corollary of Herstein's **(7)**.

COROLLARY. *If at least one zero of a polynomial $p(X) \in \Delta[X]$ lies in D but outside of Δ'', then $p(X)$ has infinitely many zeros in D.*

The next theorem provides another generalization of Herstein's theorem.

THEOREM 2. *Let D be a non-commutative division ring with centre C, and let Δ be a proper division subring which contains C, and has finite dimension d over C. Then any element of D which lies outside of Δ possesses infinitely many conjugates with respect to Δ.*

Proof. H. Cartan **(4)** has shown under the hypotheses of the theorem that D has finite dimension over Δ' equal to d, and moreover, that $\Delta'' = \Delta$. Now Δ' cannot be finite. Otherwise D is an algebraic division algebra over a finite field, and hence, by Jacobson's theorem **(9**, Theorem 8**)**, D is commutative, contrary to hypothesis. Theorem 1 now applies.

2. Isomorphic division subrings. Let Δ be a division subring of a non-commutative division ring D, such that Δ is not contained in the centre C of D. Then the number of distinct isomorphisms of Δ of the form $x^{-1}\Delta x$ is equal to the index of ∇^* in D^*, where ∇ is the centralizer of Δ in D. That this index is always infinite can be obtained from the case $A = D$ of the next proposition, as well as from Proposition 1.

PROPOSITION 2. *Let D be a division ring, and let A be a division subring with infinitely many elements. Let Δ be a division subring of D not contained in the centralizer A' of A. Then there exist infinitely many isomorphisms of Δ of the form $x^{-1}\Delta x$, with $x \in A$.*

Proof. By Theorem 1, any θ in D not in A' is moved infinitely many times by the inner automorphisms of D effected by the elements of A. Since Δ is not contained in A', there must be infinitely many distinct isomorphisms of the form $x^{-1}\Delta x$, with $x \in A$.

Let Δ be a proper division subring not contained in the centre C of a non-commutative division ring D. Let

$$H(\Delta) = \{x \in D^* | x\Delta x^{-1} = \Delta\},$$

and consider the two conditions:

(1) *index of $H(\Delta)$ in $D^* = m < \infty$;*
(2) *index of ∇^* in $H(\Delta) = r < \infty$;*

where ∇ is the centralizer of Δ in D. I wish to prove that (1) cannot hold. Since Proposition 1 asserts that (1) and (2) cannot hold simultaneously, it will be useful to note some conditions under which (2) holds. Equation (2) can

be interpreted as follows: *The inner automorphisms of D induce only finitely many distinct automorphisms in Δ.* This case certainly occurs when Δ is a field having finite degree over $\Delta \cap C$. More generally, since the centre Z of Δ is mapped onto Z by every automorphism of Δ, it is easily seen that $H(Z) \supseteq H(\Delta)$. Thus if Z is not contained in C, the case just mentioned for fields having finite degree produces the next lemma.

LEMMA 1. *Let Δ be a proper division subring of a non-commutative division ring D such that the centre Z of Δ has finite degree $n > 1$ over $Z \cap C$, where C denotes the centre of D. Then $H(\Delta)$ has infinite index in D^*.*

The proposition below is actually a result of Brauer's **(3)**. It asserts that in general h and $h + 1$ cannot both belong to $H(\Delta)$. I include the proof for the sake of completeness.

PROPOSITION 3. *Let h and $h + 1$ be non-zero elements of D. Then both h and $h + 1$ belong to $H(\Delta)$, where Δ is a division subring of D, if and only if h lies in Δ, or in the centralizer of Δ.*

Proof. The sufficiency is evident. Let $\delta \in \Delta$. Then following Brauer **(3)**;

(i) $h\delta = \delta_0 h$, where $\delta_0 = h\delta h^{-1} \in \Delta$;

(ii) $(h + 1)\delta = \delta_1(h + 1)$, where $\delta_1 = (h + 1)\delta(h + 1)^{-1} \in \Delta$.

From (ii), $h\delta + \delta = \delta_1 h + \delta_1$, so that by (i), $(\delta_0 - \delta_1)h = (\delta_1 - \delta)$. If for some choice of δ, $\delta_0 \neq \delta_1$, then $h = (\delta_0 - \delta_1)^{-1}(\delta_1 - \delta)$ lies in Δ. Otherwise, for all δ, $\delta_0 = \delta_1$. Then from (ii) $\delta = \delta_0$. This is true for all δ. Thus h lies in the centralizer of Δ. This completes the proof.

Now let h and h_0 both belong to $H(\Delta)$. Then if $h + h_0 \in H(\Delta)$, it is necessary that $hh_0^{-1} + 1 \in H(\Delta)$. By the preceding proposition hh_0^{-1} lies in Δ, or Δ'. If further $h_0 \in \Delta \cap \Delta'$, then h lies in Δ, or Δ'. This produces the

COROLLARY. *If h, h_0, and $h + h_0$ all belong to $H(\Delta)$, Δ as in the preceding proposition, then hh_0^{-1} lies in either Δ, or Δ'. If further h_0 lies in the centre of Δ, then h lies in Δ, or Δ'.*

Let D be a non-commutative division ring with centre C, and let Δ be a proper division subring not contained in C. Then there exist two elements v and d, v in D but outside of Δ, and d in Δ, such that $vd \neq dv$. Now Nagahara **(12**, Lemma 1**)** has shown that there is at most one c in $d' \cap \Delta$ such that $(v + c)d(v + c)^{-1}$ lies in Δ. Now let $(v + c)d(v + c)^{-1}$ be outside of Δ, $c \in d' \cap \Delta$. Then $v + c$ does not belong to $H(\Delta)$. It is natural to inquire whether there exist at most two c's in $d' \cap \Delta$, say c_1 and c_2, such that $v + c_1$ and $v + c_2$ belong to the same right coset of $H(\Delta)$ in D^*. This question is answered in the affirmative below.

PROPOSITION 4. *Let D be a non-commutative division ring with centre C, and let Δ be a proper division subring not contained in C. Choose v in D outside of*

Δ, and d in Δ, such that $vd \neq dv$. Let $\{c_k\}$ be a sequence of distinct elements of $d' \cap \Delta$. Then at most two elements of the sequence $\{v + c_k\}$ can belong to the right coset of $H(\Delta)$ in D^* determined by any one of them.

Proof. Suppose $v + c_k$, $k = 1, 2, 3$, all belong to the same right coset of $H(\Delta)$ in D^*, where the c_k, $k = 1, 2, 3$, are distinct elements of $d' \cap \Delta$. Then $(v + c_1)(v + c_2)^{-1} = h$, and $(v + c_3)(v + c_2)^{-1} = h_0$, where h and h_0 belong to $H(\Delta)$. This implies that $(1 - h)v = hc_2 - c_1$, and $(1 - h_0)v = h_0c_2 - c_3$. Thus,

$$(\alpha) \quad v = -c_2 + (1 - h)^{-1}(c_2 - c_1),$$

and,

$$(\beta) \quad v = -c_2 + (1 - h_0)^{-1}(c_2 - c_3).$$

Moreover, by equating v in (α) and (β), one obtains:

$$(1 - h)(1 - h_0)^{-1} = (c_2 - c_1)(c_2 - c_3)^{-1} = d_0 \in d' \cap \Delta.$$

Therefore,

$$(\gamma) \quad d_0h_0 = h + (d_0 - 1).$$

From the corollary to Proposition 3, it follows, since neither d_0 nor $d_0 - 1$ equals zero, that $h(d_0 - 1)^{-1} \in \Delta$, whence $h \in \Delta$, or else $h(d_0 - 1)^{-1} \in \Delta'$. Now h cannot belong to Δ, otherwise by (α), v must belong to Δ, contrary to its choice. Consequently $h(d_0 - 1)^{-1} \in \Delta'$, so that h belongs to the division ring A generated by Δ' and the elements c_1, c_2, and c_3. But then (α) shows that $v \in A$. Thus $v \in d'$, that is, $vd = dv$, contrary to its choice. This completes the proof.

Evidently from this proposition, $H(\Delta)$ has infinite index in D^* provided only that Δ is a proper subring of D not contained in the centre such that for some choice of d in Δ, $d \notin C$, the division ring $d' \cap \Delta$ is infinite. Otherwise every $d \in \Delta$ belongs to a finite division ring. Thus (directly, even without applying Wedderburn's theorem) Δ is an algebraic division algebra over the finite field $Z = \Delta \cap \Delta'$. Then Jacobson's theorem (9, Theorem 8) implies that $\Delta = Z$, that is, Δ is commutative, so that $d' \cap \Delta = \Delta$ for each $d \in \Delta$. Consequently Δ must be a finite field of necessarily finite degree $n > 1$ over the subfield $\Delta \cap C$. The first lemma now may be applied to complete the proof of the next theorem.

THEOREM 3. *Let D be a non-commutative division ring, and Δ a proper division subring not contained in the centre. Then there exist infinitely many distinct subrings $x\Delta x^{-1}$.*

3. Applications. Let D be a non-commutative division ring, and let Δ and A be division subrings such that the following conditions are satisfied:

(1) Δ does not contain A.
(2) A' does not contain Δ.

When A is infinite, (2) in conjunction with Proposition 2 implies that Δ has infinitely many isomorphisms of the form $a\Delta a^{-1}$ with $a \in A$. Then it is interesting to ask: Are there infinitely many different subrings $a\Delta a^{-1}$ with $a \in A$? Theorem 3 shows that the answer to this question is yes in case A contains Δ properly, inasmuch as (2) implies that Δ is not contained in the centre of A. This is a special case of (I) of the next corollary.

COROLLARY 1. *Let D be a non-commutative division ring, and let Δ and A be division subrings such that* (1) *and* (2) *above hold. Then D contains infinitely many different subrings of the form $a\Delta a^{-1}$ with $a \in A$, provided any one of the following conditions are satisfied:*

(I) *$\Delta \cap A$ is not contained in the centre of A.*
(II) *$Z \cap A$ is infinite, where Z is the centre of Δ.*
(III) *D has characteristic* 0.
(IV) *D is algebraic over the prime subfield.*

Proof. (I) Let $B = \Delta \cap A$, and let $H_B = H(B) \cap A^*$, $H_\Delta = H(\Delta) \cap A^*$. It is easily verified that $H_B \supseteq H_\Delta$. Since $B \neq A$, and since B is not contained in the centre of A, H_B, and *a fortiori* H_Δ, has infinite index in A^*. This completes the proof of (I).

(II) Since $A' \cap \Delta \neq \Delta$, and since $\Delta \cap A \neq A$, one can choose $d \in \Delta$, $d \notin A'$, and $v \in A$, $v \notin \Delta$ such that $vd \neq dv$. Now $d' \cap \Delta \cap A$ contains $Z \cap A$. By Proposition 4 the sequence $\{(v + c)\Delta(v + c)^{-1}\}$ is infinite, $c \in Z \cap A$. This completes the proof of (II).

(III)-(IV). Let P denote the prime subfield of D. If D has characteristic 0, then P is infinite, and so is $Z \cap A$. Hence (II) applies. If D is algebraic over P, then Jacobson's theorem shows that P must be infinite.

COROLLARY 2. *Let D be a non-commutative division ring, and let F be a division subring whose centralizer F' is not a field. Let d be any element of F' not contained in its centre. Let R be any division subring of F, and let $\Delta = R(d)$ denote the division subring generated by R and d. Then there exist infinitely many different $x\Delta x^{-1}$ with $x \in F'$.*

Proof. Let $A = F'$. It is clear that (1) of Corollary 1 holds inasmuch as Δ is contained in the centralizer d' of d, whereas A is not. Moreover, since $d \in \Delta \cap A$, it follows that $\Delta \cap A$ is not contained in the centre of A. Thus (I) and (2) of Corollary 1 hold, so that its conclusion applies.

It is an interesting consequence of this result that the extension D/F of the corollary possesses infinitely many intermediate division rings $x\Delta x^{-1}$ in the case $R = F$. That the hypothesis on F' is necessary in some cases for this situation to arise can be seen as follows: Let F be a division subring of D containing C, and having finite dimension over C. It is known (**10**, p. 165) that there is a $1 - 1$ correspondence between intermediate division subrings of F/C and those of D/F', and that $F'' = F$. Now suppose that

F' is a field. Then, since $F \supseteq F'$, F' has finite degree over C. If further we assume that F'/C is separable, then this extension contains only finitely many intermediate fields. Then D/F contains only finitely many intermediate division rings.

The Cartan-Brauer-Hua theorem has been generalized extensively to simple and other rings **(1, 2, 6, 14)**. I have obtained an analogue of Theorem 3 for these rings, and this has been announced in **(5)**. The new results in **(5)** neither depend upon, nor contain, the results of the present paper.

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ON THE MT^*- AND λ-CONJUGATES OF $\mathfrak{L}^\lambda$ SPACES

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1. Introduction. Marston Morse and William Transue **(9, 10)**, motivated by their theory of bilinear functions, introduced and studied vector function spaces called MT-spaces for which each element of the dual is represented by an integral with respect to a suitable (C) measure. In this paper the definition of real MT-spaces is generalized to give spaces, called MT^*-spaces, for which part but not all of the dual is of integral type and this part is called the *MT^*-conjugate* of the space. In the theory of $\mathfrak{L}^\lambda$ spaces **(6)** a conjugate space is also defined. It will be called the λ-conjugate below. An $\mathfrak{L}^\lambda$ space is an MT^*-space if and only if it contains $\mathfrak{K}$, the space of all continuous functions with compact support. The purpose of this paper is to compare the MT^*- and λ-conjugates of $\mathfrak{L}^\lambda$ spaces that are MT^*-spaces .

When E is countable at infinity the MT^*-conjugates and λ-conjugates coincide. Conditions ensuring that the MT^*-conjugate contains the λ-conjugate are given in Theorem 3.1, that the λ-conjugate includes the MT^*-conjugate in Theorems 4.1 and 4.2. Examples are given of $\mathfrak{L}^\lambda$ spaces, including the space $\bar{\mathfrak{L}}^1$ **(4**, p. 13**)**, for each of which the λ-conjugate strictly contains the MT^*-conjugate. Theorem 4.2 shows that, for a class of spaces E more general than the spaces E that are countable at infinity, the λ-conjugate always contains the MT^*-conjugate. Whether or not this is true for all E is not known. This paper makes essential use of many of the results of references **(3, 4)** and **(8)**. The author wishes to express his thanks to Professor Morse for making available pre-publication copies of **(8)** and **(10)**.

2. MT^*- and $\mathfrak{L}^\lambda$-spaces and their conjugates. Let E be an arbitrary locally compact space and let $\mathbf{R}^E$ denote the space of real valued functions on E.

Definition 2.1. We call A a (real) MT^*-space if: (i) A is a vector subspace of $\mathbf{R}^E$, (ii) A contains $\mathfrak{K}$, (iii) A contains $|x|$ if it contains x and (iv) there is a non-trivial, monotone semi-norm $\mathfrak{N}^A$ defined on A.

We note that a real MT^*-space satisfies condition (i) in the definition of real MT-spaces apart from the requirement that $\mathfrak{K}$ be dense in A **(9**, p. 168**)**. If A' denotes the topological dual of A then, as in **(9**, Corollary 10.1**)**, for every $\Phi \in A'$, the restriction of Φ to $\mathfrak{K}$ defines a Radon measure ϕ on E such that for every $f \in \mathfrak{K}$,

$$\Phi(f) = \int f d\phi. \tag{2.1}$$

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The set of elements Φ in A' for which (2.1) holds for every $f \in A$ will be called the MT^*-conjugate of A and denoted by A^*. The mapping $\Phi \to \phi$ of A' into the vector space of all Radon measures on E is an isomorphism defining the MT^*-measure conjugate $\mathfrak{A}^*$ of A (cf. **9**, p .169). If $\phi \in \mathfrak{A}^*$, every $f \in A$ is integrable (ϕ) and to $\Phi(f)$, $\Phi \in A^*$. The definition

$$||\phi||_{\mathfrak{A}^*} = \sup_{x \in A, \mathfrak{N}^A(x) > 0} \frac{|\int x d\phi|}{\mathfrak{N}^A(x)},$$

for all $\phi \in \mathfrak{A}^*$, gives $||\phi||_{\mathfrak{A}^*} = ||\Phi||_{A'}$.

A real MT-space is an MT^*-space for which $A' = A^*$. Conversely, if A is an MT^*-space for which $A' = A^*$, it is a real MT-space. Condition (ii) for MT-spaces is then satisfied by hypothesis. If $\mathfrak{K}$ is assumed to be non-dense in A then (**1**, Lemme, p. 57) implies the existence of $\Phi \in A'$, not the zero element, vanishing in $\mathfrak{K}$. The corresponding ϕ is then the zero measure so that for some $f \in A$, $0 = \int f d\phi \neq \Phi(f)$ contradicting the fact that $A' = A^*$. Examples 3.1 and 3.2 of (**10**) show that there are MT^*-spaces in which $\mathfrak{K}$ is dense that are not real MT-spaces.

Definition 2.2 (**11**, p. 53). *Let $\mathfrak{M}$, $\mathfrak{M}'$ be families of subsets of E closed under the formation of countable unions and complements where $\mathfrak{M}'$ is a proper subfamily of $\mathfrak{M}$ and has the additional property that $M \in \mathfrak{M}'$, $A \in E$, $A \subset \mathfrak{M}$ imply that $A \in \mathfrak{M}'$. For an $\mathfrak{M}$-measurable real valued function $f(P)$ on E, let $||f||_\infty$, the* ess sup *of $|f(P)|$, be the infimum of the set of numbers α such that $E_\alpha = (P:|f(P)| < \alpha)$ is in $\mathfrak{M}'$, if this set is non-void. Otherwise let $||f||_\infty$. Let $\mathfrak{L}_\infty = \mathfrak{L}_\infty(E, \mathfrak{M}, \mathfrak{M}')$ denote the subspace of $\mathbf{R}^E$ for which $||f||_\infty$ is defined and finite.*

The space $\mathfrak{L}_\infty$ is an MT^*-space with semi-norm $||\cdot||_\infty$ if every continuous function with compact support is $\mathfrak{M}$-measurable and in particular if $\mathfrak{M}$ contains all the relatively compact Borel sets on E. The elements of A' to which correspond finitely additive measures that are not countably additive are not in A^*. If $E = (0, 1)$, if $\mathfrak{M}$ denotes the Borel sets on E and $\mathfrak{M}'$ the Borel sets of the first category on E, $\mathfrak{L}_\infty$ is an MT^*-space for which A^* reduces to the zero element of A' (**2**, Corollary 1, p. 186).

If A is an MT^*-space, $\mathfrak{N}^A$ the semi-norm on A, $\mathfrak{N}^A$ will be called reflexive if

$$\mathfrak{N}^A(x) = \sup\left(\left|\int x d\phi\right| ; ||\phi||_{\mathfrak{A}^*} < 1, \phi \in \mathfrak{A}^*\right). \tag{2.2}$$

The left side of (2.2) is never less than the right side. When $\mathfrak{N}^A$ is reflexive and the supremum in (2.2) can be replaced by $\sup \int |x| d|\phi|$, for all $\phi \in \mathfrak{A}^*$ with $||\phi||_{\mathfrak{A}^*} < 1$, $\mathfrak{N}^A$ has a natural extension to all of $\bar{\mathbf{R}}^E$. If A is an MT-space $\mathfrak{N}^A$ is always reflexive and has such an extension (**9**, § 11). For the example above with $A^* = 0$, $\mathfrak{N}^A$ is not reflexive.

Length functions and the corresponding function spaces were introduced in terms of non topological spaces E and general measures in (**6**). In this

paper we suppose given a positive Radon measure μ on an arbitrary locally compact space E. If $\lambda(f)$ is defined and $0 \leq \lambda(f) \leq \infty$ for every μ-measurable function $f(P)$ with $0 \leq f(P) \leq \infty$ almost everywhere, λ is called a length function if:

(L 1) $\lambda(f) = 0$ whenever f is μ-negligible,
(L 2) $\lambda(f) \leq \lambda(g)$ whenever $f(P) \leq g(P)$ for all $P \in E$,
(L 3) $\lambda(f + g) \leq \lambda(f) + \lambda(g)$,
(L 4) $\lambda(kf) = k\lambda(f)$ for all $k > 0$,
(L 5) $f_1(P) \leq f_2(P) \leq \ldots$ for all P implies that $\lambda(\sup f_n) = \sup \lambda(f_n)$.

A length function λ will be called continuous at infinity if, for every f,

(L 6) $\lambda(f) = \sup_K \lambda(f\chi_K)$,

for all compact sets $K \in E$, where χ_K denotes the characteristic function of K. If $|f(P)|$ is μ-measurable, $\lambda(f)$ will mean $\lambda(|f|)$. A function f (set B) will be called λ-negligible if $\lambda(f) = 0$ ($\lambda(\chi_B) = 0$). $\mathfrak{L}^\lambda = \mathfrak{L}^\lambda(E, \mu)$ will denote the vector subspace of $\mathbf{R}^E$ consisting of all μ-measurable $f \in \mathbf{R}^E$ with $\lambda(f) < \infty$. $\mathfrak{L}^\lambda$ will be an MT^*-space if and only if it contains $\mathfrak{K}$. $L^\lambda = L^\lambda(E, \mu)$ will denote the normed space associated with $\mathfrak{L}^\lambda$. Every space L^λ is a Banach space (**6**, Theorem 3.1).

For every length function λ a λ-conjugate length function λ^* is defined by

$$\lambda^*(g) = \sup \int f(P)g(P)d\mu \leq \infty, \tag{2.3}$$

the supremum being taken for all $f \in \mathfrak{L}^\lambda$ with $\lambda(f) \leq 1$. The space $\mathfrak{L}^{\lambda^*}$ will be called the λ-conjugate of $\mathfrak{L}^\lambda$ and L^λ. A length function is reflexive if $\lambda(f) = \lambda^{**}(f)$ for every non-negative μ-measurable f. Necessary and sufficient conditions for the reflexivity of λ are given in (**7**, (4.1)) (for a general measure space). It can be shown that when λ is reflexive and $\mathfrak{L}^\lambda$ is an MT^*-space for which the MT^*-conjugate contains the λ-conjugate then λ is also reflexive as a semi-norm on $\mathfrak{L}^\lambda$, $|\phi| \in \mathfrak{A}^*$ if $\phi \in \mathfrak{K}^*$ and

$$\lambda(f) = \sup \left(\int |f|d|\phi|;\ ||\phi||_{\mathfrak{A}^*} \leq 1,\ \phi \in \mathfrak{A}^* \right)$$

permitting a natural extension of λ to all of $\bar{\mathbf{R}}_+{}^E$.

Let μ denote a positive Radon measure on E. For $1 \leq p < \infty$, $\mathfrak{N}_p(f) = (\int f^p d\mu)^{1/p}$ is defined and non-negative for every μ-measurable f that is defined and non-negative almost everywhere. $\mathfrak{N}_p$ then defines a length function. (L 1) follows from (**3**, Théorème 1, p. 119), (L 2) from (**3**, Proposition 10, p. 109) and (L 5) from (**3**, Théorème 3, p. 110) all applied to f^p, and (L 3) and (L 4) from (**3**, Proposition 2, p. 127). The μ-negligible and $\mathfrak{N}_p$-negligible sets coincide. If E is countable at infinity λ will be continuous at infinity. If E is arbitrary the length function $\mathfrak{N}_p$ will not be continuous at infinity if E contains a locally negligible set that is not μ-negligible. By (**3**, Théorème 5, p. 194) the spaces $\mathfrak{L}^p$ and $\mathfrak{L}^\lambda$ with $\lambda = \mathfrak{N}_p$ coincide.

For every μ-measurable $f(P)$, with $0 < f(P) < \infty$ almost everywhere, write

$$\overline{\mathfrak{N}}_p(f) = \sup_K \mathfrak{N}_p(f\chi_K),$$

where K runs through the compact subsets of E. It is easily verified that $\overline{\mathfrak{N}}_p$ is a length function that is continuous at infinity and that the $\overline{\mathfrak{N}}_p$-negligible sets are the locally negligible sets of E. We write $\overline{\mathfrak{L}}^p$ and $\bar{L}^p$ for the spaces $\mathfrak{L}^\lambda$, L^λ with $\lambda = \overline{\mathfrak{N}}_p$. From (**4**, Proposition 7, p. 13) it follows that $\overline{\mathfrak{L}}^1$ is the space of essentially integrable functions for μ.

LEMMA 2.1. *If $f \in \overline{\mathfrak{L}}^p$, the set $E(f) = (P{:}f(P) \neq 0)$ is the union of a countable sequence of compact sets and a locally negligible set E_0.*

THEOREM 2.1. *If every locally null (μ) subset of E is μ-negligible, in particular if E is countable at infinity, $\mathfrak{L}^p$ and $\overline{\mathfrak{L}}^p$, $1 \leq p < \infty$ coincide. If E contains a locally null subset that is not μ-negligible, $\overline{\mathfrak{L}}^p$ strictly contains $\mathfrak{L}^p$, $\bar{L}^p$ and L^p, $1 \leq p < \infty$, are equivalent.*

Lemma 2.1 is the analogue of (**3**, Lemme 2, p. 194). It is implied by (**4**, Corollaire, p .13) for $p = 1$. Theorem 2.1 for $p = 1$ is a consequence of results in (**4**, § 2). In both cases the extension to all p, $1 < p < \infty$ is not difficult.

When $p = \infty$ and $\mathfrak{M}$ denotes the μ-measurable subsets of E, two length functions are obtained from Definition 2.2 by taking: (1) $\lambda = \mathfrak{N}_\infty = \|\cdot\|_\infty$ with $\mathfrak{M}'$ the μ-negligible subsets of E; and (2) $\lambda = \overline{\mathfrak{N}}_\infty = \|\cdot\|_\infty$ with $\mathfrak{M}'$ the locally negligible subsets of E. $\overline{\mathfrak{N}}_\infty$ is continuous at infinity, $\mathfrak{N}_\infty$ is not if E contains a locally negligible set that is not μ-negligible. The $\mathfrak{L}^\lambda$ spaces $\mathfrak{L}_\infty(E, \mathfrak{M}, \mathfrak{M}')$ corresponding to (1) and (2) respectively will be denoted below by $\mathfrak{L}^\infty$ and $\overline{\mathfrak{L}}^\infty$. These spaces are MT^*-spaces. Since $\overline{\mathfrak{N}}_\infty(f) \leq \mathfrak{N}_\infty(f)$, $\mathfrak{L}^\infty$ is always contained in $\overline{\mathfrak{L}}^\infty$. In contrast to the case with $p < \infty$, L^∞ and $\bar{L}^\infty$ need not coincide. For example, if E contains a locally negligible set D with $\mu^*(D) = \infty$, $f_a(P) = a\chi_D$ is in $\mathfrak{L}^\infty$ for every finite a and the equivalence classes $\hat{f}_a$ in L^∞ are different for different positive values of a. Each f_a is in $\overline{\mathfrak{L}}^\infty$ but all belong to the equivalence class of $g(P) \equiv 0$. We note that the dual of $\mathfrak{L}^1$ is $\overline{\mathfrak{L}}^\infty$.

3. The λ-conjugate of $A = \mathfrak{L}^\lambda$. In the sequel E will denote an arbitrary locally compact space, μ a positive Radon measure on E, K, K_i compact subsets of E, and λ will be an arbitrary length function for which $\mathfrak{L}^\lambda$ contains $\mathfrak{K}$. $\lambda(B)$ will be an abbreviation for $\lambda(\chi_B)$. A function g is locally integrable if it is μ-measurable and if $g\chi_K$ is integrable (μ) for every K. (Since E is locally compact this is equivalent to the definition in (**8**)).

When $\mathfrak{L}^\lambda$ is an MT^*-space it contains the characteristic function of every relatively compact μ-measurable subset of E. $\mathfrak{L}^\lambda$ contains $\mathfrak{K}$ and since, given K, there exists a continuous function $f \in \mathfrak{K}$ coinciding with χ_K in K, if $B \subset K$ is μ-measurable $\lambda(B) \leq \lambda(K) \leq \lambda(f) < \infty$ and $\chi_B \in L^\lambda$. It then follows from

(2.3) that every $g \in \mathfrak{L}^{\lambda *}$ is locally integrable. Thus g defines a Radon measure $g \cdot \mu$ (8, § 1) with values

$$\int f\, d(g \cdot \mu) = \int fg\, d\mu = g(f), g \in (L^\lambda)', \tag{3.1}$$

for all $f \in K$. If for every $g \in \mathfrak{L}^{\lambda *}$, (3.1) extends to all $f \in \mathfrak{L}^\lambda$, the λ-conjugate of $\mathfrak{L}^\lambda$ is contained in the MT^*-conjugate. Since the right side of (3.1) is finite for all $f \in \mathfrak{L}^\lambda$ if $g \in \mathfrak{L}^{\lambda *}$, (8, Theorem 1.1) shows that it is sufficient to show that the left side is also always finite. There is no loss of generality in assuming that g is positive (that is non-negative) so that $g \cdot \mu$ is a positive Radon measure on E.

LEMMA 3.1. *If $f \in \bar{\mathbf{R}}_+^E$ is measurable (μ) and $g > 0$ is locally integrable and if $X = \cup_1^\infty K_i$, then*

$$\int^* fg\chi_X d\mu = \int^* f\chi_X d(g \cdot \mu) < \infty, \tag{3.2}$$

and both are finite if $f \in \mathfrak{L}^\lambda$ and $g \in \mathfrak{L}^{\lambda *}$.

Proof. The equality (3.2) is a corollary of (8, Lemma 5.1). When $f \in \mathfrak{L}^\lambda$ and $g \in \mathfrak{L}^{\lambda *}$, $\int^* fg\chi_X d\mu < \int fg\, d\mu < \infty$.

LEMMA 3.2. *If $f \in \mathfrak{L}^\lambda$ is a positive lower semi-continuous function and $g \in \mathfrak{L}^{\lambda *}$ then* (3.1) *holds.*

Proof. There is no loss of generality in supposing that g is non-negative. Then, using (3.1),

$$\int f\, d(g \cdot \mu) = \sup_{\substack{h \in \mathfrak{K} \\ 0<h<f}} \int h\, d(g \cdot \mu) = \sup_{\substack{h \in \mathfrak{K} \\ 0<h<f}} \int hg\, d\mu < \int fg\, d\mu < \infty.$$

Note. Lemmas 3.1 and 3.2 are also a consequence of (4, (Propositions 2 and 3, p. 9, and Theorem 1, p. 43)). See also (8, Note, p. 478).

LEMMA 3.3. *If $g \in \overline{\mathbf{R}}_+^E$ is locally integrable every μ-negligible set is $g \cdot \mu$-negligible.*

Proof. Suppose that B is μ-null. If $g \in \overline{\mathbf{R}}_+^E$, $\int g\chi_B\, d\mu = 0$ by (3, Théorème 1, p. 119). Given $\epsilon > 0$, there exists an open set $U \supset B$ with $\mu(U) < \frac{1}{2}\epsilon$ and a l.s.c. function $h > \chi_B g$ with $\mu(h) < \frac{1}{2}\epsilon$. Then env. sup. (χ_U, h) is l.s.c., $> \chi_U g$ and

$$\int^* \chi_U g\, d\mu < \int^* h\, d\mu + \int^* \chi_U d\mu < \epsilon.$$

If g is locally integrable, the measure $g \cdot \mu$ is defined > 0 and, by (3, Corollaire 4, p .158), and Lemma 3.1,

$$(g\,.\,\mu)^*(B) \le (g\,.\,\mu)^*(U) = \sup_{K\subset U} g\,.\,\mu(K) = \sup_{K\subset U} \int \chi_K d(g\,.\,\mu)$$
$$= \sup_{K\subset U} \int \chi_K g\, d\mu \le \int \chi_U g\, d\mu < \epsilon.$$

Since ϵ is arbitrary B is $g\,.\,\mu$-null.

THEOREM 3.1. *Let $g \in \mathfrak{L}^{\lambda*}$. Then* (i) *$\hat{g}$ is in the λ-conjugate $\mathfrak{L}^{\lambda*}$ and is in the MT^*-conjugate if and only if $\int f\, d(g\,.\,\mu) = 0$ for every $f \in \mathfrak{L}^{\lambda}$ for which fg vanishes in E and* (ii) *if the MT^*-conjugate does not contain the λ-conjugate, E contains a locally negligible set B with $\lambda(B) < \infty$, with $\lambda(U) = \infty$ for every open set $U \supset B$ and $\mathfrak{L}^{\lambda*}$ contains g with $g\chi_B = 0$ and with $(g\,.\,\mu)^*(B) = \infty$.*

Proof. (i) Suppose that $g \in \mathfrak{L}^{\lambda*}$ is positive. If $f \in \mathfrak{L}^{\lambda}$, $\int fg\, d\mu < \infty$ and, for fixed f, the set where $fg \neq 0$ is the union of X, the union of a sequence of compact sets, and a μ-negligible set E'. If $E_0 = (P{:}fg(P) = 0)$, $E = E_0 \cup E' \cup X$ and E_0 is measurable (μ). Suppose that f is non-negative. Using Lemma 3.1,

$$\int fg\, d\mu = \int fg\chi_X d\mu = \int f\chi_X d(g\,.\,\mu) \le \int^* f\, d(g\,.\,\mu)$$

and, using Lemma 3.3, we have

$$\int^* f\, d(g\,.\,\mu) \le \int^* f\, \chi_{E'} d(g\,.\,\mu) + \int^* f\, \chi_{E_0} d(g\,.\,\mu) + \int^* f\, \chi_X d(g\,.\,\mu)$$
$$= \int^* f\, \chi_{E_0} d(g\,.\,\mu) + \int fg\, d\mu.$$

Equality then holds if

$$\int^* f\, \chi_{E_0} d(g\,.\,\mu) = 0.$$

Conversely, if $g \in \mathfrak{L}^{\lambda*}$ and if $\hat{g}$ is in the MT^*-conjugate, then, whenever $f \in \mathfrak{L}^{\lambda}$ and fg vanishes in E, $\int f\, d(g\,.\,\mu) = \hat{g}(f) = \int fg\, d\mu = 0$. When one or both of f, g is not positive the extension is trivial.

(ii) Suppose that $\mathfrak{L}^{\lambda*}$ contains g with $\hat{g}$ not in the MT^*-conjugate A^*. Since g^+ and g^-, the positive and negative parts of g are also in $\mathfrak{L}^{\lambda*}$ and $\hat{g}^+$ and $\hat{g}^-$ in A^* would imply that $\hat{g}$ was in A^*, there is no loss of generality in assuming that g is positive. There then exists a positive $f \in \mathfrak{L}^{\lambda}$ with $fg(P) = 0$ in E but with $\int^* f\, d(g\,.\,\mu) > 0$ and, since $\int fg\, d\mu = 0$ (**8**, Lemma 2.1) implies that $\int^* f\, d(g\,.\,\mu) = \infty$,

Let $E_n = (P{:}f(P) > 1/n)$. The μ-measurability of f implies that E_n is μ-measurable. Since $\chi_{E_n} \le n f(P)$, $\lambda(E_n) \le n\lambda(f)$ for each n. Furthermore

$$\int \chi_{E_n} g\, d\mu \le n \int fg\, d\mu = 0, \qquad n = 1, 2, \ldots,$$
$$\sup_{K\subset E} \int \chi_{E_n \cap K} d(g\,.\,\mu) = \sup_{K\subset E} \int \chi_{E_n \cap K}\, g\, d\mu = 0,$$

so that each E_n is locally negligible.

Let $f_n(P) = \min. (n, f(P))$. Then f_n increases to f and by **(3**, Théorème 3, p. 110**)**,

$$\sup_n \int^* f_n \chi_{E_n} d(g \,.\, \mu) = \int^* f\, d(g \,.\, \mu) = \infty.$$

Thus, for all sufficiently large n,

$$\int^* f_n \chi_{E_n} d(g \,.\, \mu) > 0,$$

$$\int^* \chi_{E_n} d(g \,.\, \mu) \geqslant n^{-1} \int^* f_n \chi_{E_n} d(g \,.\, \mu) > 0,$$

and **(8**, Lemma 2.1**)** implies that

$$\int^* \chi_{E_n} d(g \,.\, \mu) = \infty. \tag{3.3}$$

Finally, for all n for which (3.3) holds, $\lambda(U) = \infty$ for every open set U containing E_n since otherwise Lemma 3.2 implies that

$$\int^* \chi_{E_n} d(g \,.\, \mu) \leqslant \int \chi_U d(g \,.\, \mu) = \int \chi_U g\, d\mu \leqslant \lambda(U)\lambda^*(g) < \infty,$$

giving a contradiction.

As a corollary we list some of the conditions that imply that the MT^*-conjugate A^* of $A = \mathfrak{L}^\lambda$ contains the λ-conjugate $\mathfrak{L}^{\lambda*}$:

(1) E is countable at infinity.

(2) For every $g \in \mathfrak{L}^{\lambda*}$, $g \,.\, \mu$ is bounded.

(3) For every $g \in \mathfrak{L}^{\lambda*}$ and every μ-measurable B, $(g \,.\, \mu)^*(B) = \infty$ implies that $\lambda(B) = \infty$.

(4) For every locally integrable g, $(g \,.\, \mu)^*(B) = \infty$, $\lambda(B) < \infty$, imply that $\lambda^*(g) = \infty$.

(5) If B is μ-measurable and if $\lambda(U) = \infty$ for every open set U containing B then $\lambda(B) = \infty$.

(6) $\lambda(E) < \infty$ or $\lambda(K)$ is bounded for all K in E.

If $A = \mathfrak{L}^p$, $1 \leqslant p < \infty$, and B is μ-measurable then $\mathfrak{N}_p(B) = \mu^{*1/p}(B)$ which implies (5). To prove (6) suppose that $\lambda(E) = M < \infty$ or that $\lambda(K) \leqslant M < \infty$ for every $K \subset E$. The existence of B with $(g \,.\, \mu)^*(B) = \infty$ implies that $(g \,.\, \mu)^*(E) = \infty$ and **(3**, Corollaire 4, p. 158**)** implies that E contains compact subsets with arbitrarily large $(g \,.\, \mu)$-measure. Then

$$\lambda^*(g) \geqslant \sup_K \int \chi_K g\, d\mu/\lambda(K) \geqslant \sup_K g \,.\, \mu(K)/M = \infty,$$

so that (4) applies. The spaces $\mathfrak{L}^\infty$, $\overline{\mathfrak{L}}^\infty$ satisfy (6).

Theorem 3.2. *There exist MT^*-spaces $\mathfrak{L}^\lambda$, with λ reflexive and continuous at infinity, for which the λ-conjugate strictly contains the MT^*-conjugate. In particular the spaces $\overline{\mathfrak{L}}^p$, $1 \leqslant p < \infty$, are of this type for suitable μ, E.*

Proof. Let E, D, $T_n(y)$ and μ be defined as in the example of a locally compact space E that is not countable at infinity given in (**3**, Exercice 4, p. 116) and let $g = \chi_{E-D}$. Then $\overline{\mathfrak{N}}_\infty(g) = 1$ and $\hat{g} \in \bar{L}^\infty = \mathfrak{L}^{\lambda *}$, $\lambda = \overline{\mathfrak{N}}_1$. As in (**8**, Exercice 4) $\chi_D \in \bar{\mathfrak{L}}^1$ but is not integrable $(g\,.\,\mu)$ so that $\hat{g}$ does not belong to the MT^*-conjugate of $\bar{\mathfrak{L}}^1$

To give an example for $1 < p < \infty$ let E, D be as before but for δ fixed, $0 < \delta < 1$, P_{ni} the point $(1 - n, i/n^2)$, define $\beta(P_{ni}) = n^{-2-\delta}$ and define $\beta(P) = 0$ for the points $(0, y)$ in E. Let μ' denote the measure determined by the masses $\beta(P)$. Define $g(P) = n^{\delta-1}$ for $P = P_{ni}$ $(i = 0, 1, \ldots, n^2;$ $n = 1, 2, \ldots;)$, $g(P) = 0$ elsewhere in E. Actual computation shows that every compact subset of E has finite μ'-measure and that, for $\lambda = \overline{\mathfrak{N}}^p$, $\lambda^*(g) = \overline{\mathfrak{N}}^q(g) < \infty$ so that $\hat{g} \in \mathfrak{L}^{\lambda *}$. Suppose that B is a subset of $(0, y)$ dense (in the usual topology on $\mathbf{R}$) on some interval (a, b), $-1 < a < b < 1$ and let $U_{B,n} = \cup\, T_i(y)$ for all $i \geqslant n$, $y \in B$. Computation shows that $(g\,.\,\mu')^*(U_{B,n}) = \infty$ for every n. As in the Bourbaki example every open set containing D contains some set $U_{B,n}$ and therefore $(g\,.\,\mu')^*(D) = \infty$ although D is locally negligible $(g\,.\,\mu')$. Since the dual of $\bar{\mathfrak{L}}^p$ coincides with the λ-conjugate, $1 < p < \infty$, the λ-conjugate strictly contains the MT^*-conjugate. The length functions $\overline{\mathfrak{N}}_p$ are continuous at infinity and reflexive.

Remark. The right side of (2.1) is $\mathfrak{N}_1(f, \phi)$. Replacing $\mathfrak{N}_1$ by $\overline{\mathfrak{N}}_1$ in the definition of the MT^*-conjugate gives a conjugate which always contains the λ-conjugate when $A = \mathfrak{L}^\lambda$.

4. The MT^*-conjugate of $A = \mathfrak{L}^\lambda$. In this section Φ denotes an arbitrary element of the MT^*-conjugate of $A = \mathfrak{L}^\lambda$, ϕ the Radon measure corresponding to Φ determined by the restriction of Φ to K.

LEMMA 4.1. *If $\Phi \in A^*$ and $A = \mathfrak{L}^\lambda(\mu)$ there exists a locally μ-integrable function g for which the measure $g\,.\,\mu$ coincides with ϕ.*

Proof. There is no loss of generality in assuming $\Phi \geqslant 0$. By hypothesis every $f \in A$ is integrable (ϕ) with

$$\Phi(f) = \int f\,d\phi. \tag{4.1}$$

For each compact set K every μ-measurable subset e is in A and $\Phi(\chi_e) = \int \chi_e d\phi = \phi(e)$. If $\mu(e) = 0$,

$$|\phi(e)| = |\Phi(e)| \leqslant |\Phi|\mathfrak{N}^A(\chi_e) = |\Phi|\lambda(e).$$

Since $\lambda(e) = 0$ whenever $\mu(e) = 0$, every set that is locally μ-negligible is locally ϕ-negligible. The Lebesgue-Nikodym theorem (**4**, Théorème 2, p. 47) then implies that ϕ is a measure of base μ, that is, that there exists a locally integrable point function g with $\phi = g.\ \mu$ (**4**, p. 42).

THEOREM 4.1. *Let $\Phi \in A^*$ and let g in $\mathbf{R}^E$ be locally integrable with $g\,.\,\mu = \phi$. Then* (i) *$g \in \mathfrak{L}^{\lambda *}$ if and only if $\mu(g\chi_B) = 0$ for every ϕ-negligible set B that is*

the union of a sequence $\{B_i\}$ *with each function* χ_{B_i} *in* $\mathfrak{L}^{\lambda}$, *and* (ii). *If* $g \notin \mathfrak{L}^{\lambda*}$, *E contains a set D, negligible* $(g \,.\, \mu)$ *and locally negligible* (μ) *with* $\lambda(D) < \infty$ *and* $\mu^*(D) = \infty$.

Proof. (i) There is no loss of generality in considering only positive f and Φ in the proof of (i) and (ii). If $f \in \mathfrak{L}^{\lambda}$, $\Phi \in A^*$, then $f \in \mathfrak{L}^1(\phi)$ and $E = E_0 \cup E' \cup X$, where $E_0 = (P{:}f(P) = 0)$, E' is ϕ-negligible and X is the union of a countable sequence of compact sets. Since E, E_0 and X are measurable (μ) so is E'. Since $\int f\chi_X d(g \,.\, \mu) \leqslant \int f\, d(g \,.\, \mu) = \int f\, d\phi < \infty$, (3.2) holds finitely. Since

$$fg\chi_{E_0}$$

vanishes,

$$\int fg\chi_{E_0} d\mu = 0.$$

Thus

$$\int f\, d\phi = \int f\, d(g \,.\, \mu) = \int fg\chi_X d\mu \leqslant \int^* fg\, d\mu \tag{4.4}$$

$$\leqslant \int^* fg\chi_{E_0} d\mu + \int^* fg\chi_{E'} d\mu + \int fg\chi_X d\mu = \int^* fg\chi_{E'} d\mu + \int f\, d\phi.$$

If $E_i' = (P \in E'{:}f(P) > 1/i)$, $E' = \bigcup_1^\infty E_i'$, each E_i' is μ-measurable and $\lambda(E_i') \leqslant i\lambda(f) < \infty$ so that each $\chi_{E_i'} \in \mathfrak{L}^{\lambda}$. If the hypothesis of (i) is satisfied, $\int \chi_{E'} g\, d\mu = 0$ which implies that $\int \chi_{E'} fg\, d\mu = 0$. Then $\int fg\, d\mu = \int f\, d\phi = \Phi(f)$ for every $f \in \mathfrak{L}^{\lambda}$ and $g \in \mathfrak{L}^{\lambda*}$. Conversely, if $g \in \mathfrak{L}^{\lambda*}$, $\int fg\, d\mu < \infty$ for every $f \in \mathfrak{L}^{\lambda}$ and, by (**8**, Theorem 1.1),

$$\int fg\, d\mu = \int f\, d(g \,.\, \mu) = \Phi(f) \tag{4.5}$$

for all $f \in \mathfrak{L}^{\lambda}$. If B_i is ϕ-negligible with $\chi_{B_i} \in \mathfrak{L}^{\lambda}$ (4.4) and (4.5) imply that

$$\int g\chi_{B_i}\, d\mu = 0$$

whence $\int g\chi_B\, d\mu = 0$ if $B = \bigcup_1^\infty B_i$.

(ii) If $g \notin \mathfrak{L}^{\lambda*}$ (i) implies that there exists a ϕ-negligible set $B = \bigcup_1^\infty B_i$, where $\lambda(B_i) < \infty$, $i = 1, 2, \ldots$, and such that $\mu^*(g\chi_B) > 0$. Then (**8**, Lemma 2.1) implies that $\mu^*(g\chi_B) = \infty$. Writing $B(n) = \bigcup_1^n B_i$, since $B(n) \uparrow B$, $\mu^*(g\chi_{B(n)})$ is positive and therefore infinite for all sufficiently large n, say $n > n_0$. We show that for a fixed $n > n_0$, $D = (P \in B(n) : g(P) > 0)$ satisfies all the conditions (ii). We note that $\lambda(D) \leqslant \lambda(B(n)) < \infty$ and that $\mu^*(g\chi_D) = \mu^*(g\chi_{B(n)}) = \infty$. Consider $g_m(P) = \min\,(m, g(P))$. It is locally integrable and defines a Radon measure $g_m \,.\, \mu$ with $0 < g_m \,.\, \mu \leqslant g \,.\, \mu$ and $g_m \,.\, \mu(D) \leqslant g \,.\, \mu(D) \leqslant g \,.\, \mu(B) = 0$. Now

$$\infty = \int^* g\chi_D d\mu = \sup_m \int^* g_m\chi_D d\mu,$$

and (**8**, Lemma 2.1) implies that $\mu^*(g_m\chi_D) = 0$ or ∞ for each m and therefore is infinite for all sufficiently large m so that

$$\int^* \chi_D d\mu \geqslant m^{-1} \int^* g_m \chi_D d\mu = \infty.$$

Finally, if $D_i = (P \in D : g(P) > 1/i)$, $D = \cup_1^\infty D_i$ and, for an arbitrary compact set K,

$$\int \chi_{D\cap K} d\mu = \sup_i \int \chi_{D_i\cap K} d\mu \leqslant \sup_i i \int g\chi_{D_i \cap K} d\mu$$
$$\leqslant \sup_i i \int \chi_D \, d(g \,.\, \mu) = 0.$$

Thus D is locally μ-negligible.

A variety of conditions, sufficient to ensure that the λ-conjugate contains the MT^*-conjugate, follow from Theorem 4.1. We mention only: (i) E is countable at infinity, (ii) μ is bounded and (iii) $\mu^*(B) = \infty$ implies that $\lambda(B) = \infty$. Condition (iii) shows that if $A = \mathfrak{L}^p$, $1 \leqslant p < \infty$, the λ-conjugate always contains the MT^*-conjugate so that these conjugates then coincide. Actually, each of (i)-(iii) implies that every locally integrable $g \in \mathbf{R}^B$ is in $\mathfrak{L}^{\lambda*}$ whereas $\mathfrak{L}^{\lambda*}$ will contain A^* if to each $\Phi \in A^*$ corresponds one $g \in \mathfrak{L}^{\lambda*}$ with $g \,.\, \mu \equiv \phi$. If g is locally integrable with $g \,.\, \mu \equiv \phi$, every g' that is locally equivalent to g is also locally integrable and $g' \,.\, \mu \equiv \phi$. Consider (**8**, Example 5.2) where $g = \chi_H$, $f = \chi_K$ and the length functions $\lambda = \overline{\mathfrak{N}}_p$, $1 \leqslant p < \infty$. Then $g \notin \mathfrak{L}^{\lambda*}$ but g is locally equivalent to the zero element of $\mathfrak{L}^{\lambda*}$. More generally, let E, D, and μ be defined as in Theorem 3.2 and let λ denote an arbitrary length function. Then D is locally negligible with $\mu^*(D) = \infty$ but every locally negligible subset of $E - D$ is μ-negligible. Replacing g by $g\chi_D$ gives a g' locally equivalent to g with $\int^* g'\chi_B \, d\mu = \int^* g\chi_{B\cap D} d\mu = 0$ for every locally negligible set B with $\lambda(B) < \infty$. Since

$$(g' \,.\, \mu)^*(B) \leqslant \int^* g'\chi_B \, d\mu$$

by (4.4) this contradicts Theorem 4.1 (ii) if g' is not in $\mathfrak{L}^{\lambda*}$.

Edwards (**5**, p. 143) defines the μ-measure of a μ-measurable set B to be $\mu(B) = \sup_K \int \chi_{B\cap K} d\mu$, where K runs through the compact subsets of E.

THEOREM 4.2. *If E contains E^* with $\mu^*(A) < \infty$ for every locally negligible set contained in $E - E^*$ and if E^* is the union of a countable collection of sets of finite μ-measure, then for every MT^*-space $\mathfrak{L}^\lambda$ on E, the λ-conjugate contains the MT^*-conjugate.*

Proof. First suppose that $\mu(E^*) < \infty$. By the argument of (**5**, Theorem 7 (4)), $E^* = Q_1 \cup Q_2$ where Q_1 is the union of a countable collection of compact sets and Q_2 is locally null. If $A \subset E^*$ with $\mu(A) = 0$, $\mu^*(A) = \infty$, $\mu(Q_1 \cap A) = 0$. Set $g' = g\chi_X$, where $X = CE^* \cup Q_1$. If $K \subset CE$ or $K \subset Q_1$, $g(P) = g'(P)$ for $P \in K$. Suppose that $\mu(K \cap CE^*) > 0$ and $\mu(K \cap E^*) > 0$.

Then $K = N \cup_1^\infty K_n$ where N is μ-negligible and each K_n is compact and contained in one of E^*, CE^* (**3**, pp. 181-2). Since $\mu(K_n \cap Q_2) = 0, n = 1, 2, \ldots,$ $g = g'$ almost everywhere in each K and g' is locally equivalent to g. For each such $A \subset E^*$,

$$\mu^*(g'\chi_A) = \mu^*(g\chi_{Q_1 \cup A}) = 0$$

contradicting Theorem 4.2 (ii) if $g' \notin \mathfrak{L}^{\lambda *}$. If $E^* = \cup_1^\infty E_n$ with $\mu(E_n) < \infty$, $n = 1, 2, \ldots,$ each $E_n = Q_{1n} \cup Q_{2n}$ as above and the preceding argument holds when applied to $\cup_n Q_{1n}$, $\cup_n Q_{2n}$ in place of Q_1 and Q_2 respectively.

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A CONVERGENCE THEOREM FOR DOUBLE L^2 FOURIER SERIES

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1. Our aim in this paper is to extend a known theorem about the convergence of subsequences of the partial sums of the Fourier series in one variable of class L^2 to Fourier series in two variables of the same class, (**1**, p. 396). The theorem asserts that for each function f in L^2, there is a sequence $\{m_\nu\}$ of positive integers of upper density one such that

$$s_{m_\nu}(x;f)$$

converges to f almost everywhere, where $s_m(x;f)$ denotes the mth partial sum of the Fourier series of f. The sequence $\{m_\nu\}$ depends on the function f but not on the point x (**3**, p. 264). The main tools of proof used were the theorem of Kolmogoroff asserting the almost everywhere convergence of lacunary subsequences of partial sums of L^2 Fourier series and the theorem of Kolmogoroff and Seliverstoff (**3**, p. 253). These same tools are available in the two-dimensional case (**2**), but they do not seem to be adequate in themselves to obtain an extension.

Our method of proof is the following. First we extend the one-dimensional theorem so that a single sequence $\{m_\nu\}$ of upper density one will serve for a given sequence $\{f_n\}$ of functions, each f_n belonging to L^2. From this we may generalize to the two-dimensional case by considering first iterated limits of partial sums.

2. The definition of upper density for a sequence $\{m_\nu\}$ of positive integers strictly increasing to ∞ is as follows. Let $\sigma(n)$ be the number of terms of the sequence less than or equal to n. We say the sequence is of upper density β if $\limsup \sigma(n)/n = \beta$.

THEOREM 1. *Let $\{f_n\}$ be a sequence of functions, each of class L^2. Then there is a sequence $\{p_\nu\}$ of upper density one such that*

$$s_{p_\nu}(x;f_n)$$

converges to f_n almost everywhere for each n.

Let $\{\lambda_r\}$ and $\{k_r\}$ be two sequences of integers each strictly increasing to ∞ and such that $k_{r+1} > 2k_r$ and $r+1$ divides k_r. At a later stage we shall

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impose a further restriction relating to the size of k_r. Let $m_k = \lambda_r^k$, $k_r < k < 2k_r$, and let

$$\sum_{m=-\infty}^{+\infty} c_m(n)e^{imx}$$

denote the Fourier series of the function $f_n(x)$. For any function $f(x)$ of L^2 with Fourier coefficients c_m, we introduce the following notation:

$$\epsilon_k = \sum_{|m|=m_k+1}^{m_{k+1}} |c_m|^2,\ k = k_r, k_r + 1 \ldots 2k_r - 1;\ D_r = \sum_{k=k_r}^{2k_r-1} \epsilon_k. \tag{1}$$

If α of the k_r numbers ϵ_k are greater than $\beta D_r/k_r$, then

$$\alpha\beta \frac{D_r}{k_r} < D_r \qquad \text{or} \qquad \alpha < \frac{k_r}{\beta}. \tag{2}$$

Now let $\epsilon_k(n)$ and $D_r(n)$ correspond to $f_n(x)$ as ϵ_k and D_r in (1) correspond to $f(x)$. Fix r, and consider $f_1, f_2, \ldots, f_r$. For each n, $n = 1, 2, \ldots, r$, at least $rk_r/(r+1)$ of the numbers $\epsilon_k(n)$ are less than $(r+1)D_r(n)/k_r$ since by (2) the number greater does not exceed $k_r/(r+1)$. There must be some k, say $k(r)$, $k_r < k(r) < 2k_r - 1$, such that

$$\epsilon_{k(r)}(n) < \frac{(r+1)D_r(n)}{k_r}, \qquad n = 1, 2, \ldots, r$$

since among the rk_r numbers $\epsilon_k(n)$, no more than $rk_r/(r+1)$ of them exceed the above. Thus for $k = k(r)$

$$\sum_{|m|=m_k+1}^{m_{k+1}} |c_m(n)|^2 \log(|m|) < 2k_r (\log \lambda_r)\epsilon_k(n) < 2(r+1)(\log \lambda_r)D_r(n), \tag{3}$$
$$n = 1, 2, \ldots, r.$$

For fixed λ_r and n,

$$\sum_{|m|=\lambda_r^k}^{\infty} |c_m(n)|^2$$

goes to 0 as k increases to ∞. We may thus choose $k = k_r$ subject to the previous conditions and so large that

$$2(r+1)(\log \lambda_r)D_r(n) < 2^{-r}, \qquad n = 1, 2, \ldots, r.$$

From (3),

$$\sum_{\substack{k=k(r)\\ r>n}} \sum_{|m|=m_k+1}^{m_{k+1}} |c_m(n)|^2 \log(|m|) < 2 \sum_{r>n} (r+1)(\log \lambda_r)D_r(n) < \infty$$

for all n. Now let $\{p_\nu\}$ take on the values m, $m_k < m < m_{k+1}$, $k = k(r)$, $r = 1, 2, \ldots$. It is easily seen that $\{p_\nu\}$ is of upper density one and the almost everywhere convergence of each sequence

$$s_{p_\nu}(x; f_n)$$

to f_n follows as in our original proof (**1**, p. 396).

Remark. For convenience we suppose that $6(r+1)$ divides k_r. Among the rk_r numbers $\epsilon_k(n)$, no more that $rk_r/3(r+1)$ of them exceed $3(r+1)D_r(n)/k_r$, for the same reasons as used above. Hence for more than one-half the indices k in the given range,

$$\epsilon_k(n) < \frac{3(r+1)D_r(n)}{k_r}, \qquad n = 1, 2, \ldots, r,$$

a fact we shall use in the proof of Theorem 2.

3. We first extend the notion of upper density for single sequences of positive integers to double sequences. Let P be a set of ordered pairs of positive integers (p, q), and let $\sigma(m, n)$ be the number of pairs (p, q) from P such that $p < m$ and $q < n$. If β is the largest number for which there are sequences $\{m_k\}$ and $\{n_k\}$ of positive integers each strictly increasing to ∞ as k goes to ∞ such that $\lim_{k\to\infty}\sigma(m_k, n_k)/m_k n_k = \beta$ we say that the set P has upper density β. It is easily seen that a largest number must exist. We may state our generalized theorem. Part (i) relates to iterated limits and is used in the proof of part (ii). Let $s_{p,q}(x, y; f)$ denote the pqth partial sum of the Fourier series of an integrable function $f(x, y)$ and let Ω be the square in the xy plane with $(0, 0)$ and $(2\pi, 2\pi)$ as opposite vertices.

THEOREM 2. *Let $f(x, y)$ belong to $L^2(\Omega)$.*

(i) *There exist sequences $\{p_\mu\}$, $\{q_\nu\}$ of positive integers, each separately of upper density one, such that almost everywhere*

$$\lim_{\mu\to\infty}\left[\lim_{\nu\to\infty} s_{p_\mu, q_\nu}(x, y; f)\right] = \lim_{\nu\to\infty}\left[\lim_{\mu\to\infty} s_{p_\mu, q_\nu}(x, y; f)\right] = f(x, y).$$

(ii) *There exists a double sequence P' of positive integers of upper density one such that almost everywhere*

$$\lim_{\substack{p, p\to\infty \\ (p, q)\in P}} s_{p,q}(x, y; f) = f(x, y).$$

In the double limit of part (ii), p and q go to ∞ independently except that the pair (p, q) must belong to P. Let $c_{m,n}$ denote the Fourier (exponential) coefficient of $f(x, y)$. Since for each n,

$$\sum_{m=-\infty}^{+\infty} |c_{m,n}|^2 < \infty,$$

the series

$$\sum_{m=-\infty}^{+\infty} c_{m,n} e^{imx}$$

is the Fourier series of a function $f_n(x)$ in $L^2(0, 2\pi)$. We have

$$c_{m,n} = \frac{1}{2\pi i}\int_0^{2\pi} e^{-imx}\left\{\frac{1}{2\pi i}\int_0^{2\pi} f(x, y)e^{-iny}\, dy\right\} dx$$

so that

$$2\pi i f_n(x) = \int_0^{2\pi} f(x, y)e^{-iny}\, dy.$$

By Theorem 1, there is a sequence $\{p_\mu\}$ of upper density one such that for almost every x and every $n = 0, \pm 1, \ldots$

$$\lim_{\mu\to\infty} \sum_{m=-p_\mu}^{p_\mu} c_{m,n}e^{imx} = f_n(x).$$

Hence, for every fixed q, every y, and almost every x

$$\lim_{\mu\to\infty} s_{p_\mu,q}(x, y; f) = \sum_{n=-q}^{q} f_n(x)e^{iny}. \tag{4}$$

By Parseval's equality

$$2\pi \sum_{n=-\infty}^{+\infty} |f_n(x)|^2 = \int_0^{2\pi} |f(x, y)|^2 dy$$

which is finite for almost every x. Hence, for almost every x, each member of the family, indexed by x, of series

$$\sum_{n=-\infty}^{+\infty} f_n(x)e^{iny} \tag{5}$$

is the Fourier series of an L^2 function of the variable y, that is, $f(x, y)$. The numbers $f_n(x)$ are then the Fourier coefficients.

Let $\{\alpha_r\}$ be a sequence of numbers, $0 < \alpha_r < 2\pi$, such that $\sum_{r=1}^{\infty}\alpha_r < \infty$. Let $\{\lambda_r\}$ and $\{k_r\}$ be two sequences of positive integers, each strictly increasing to ∞ and such that $k_{r+1} > 2k_r$. A further restriction is needed on $\{k_r\}$. For fixed λ_r, choose k_r so large that

$$\sum_{|n|>\lambda_r^k} \int_0^{2\pi} |f_n(x)|^2 dx < \frac{2^{-r-1}\alpha_r}{\log \lambda_r}, \qquad k > k_r. \tag{6}$$

This is possible since the left side of (6) equals

$$2\pi \sum_{|n|>\lambda_r^k} \sum_{m=-\infty}^{+\infty} |c_{m,n}|^2.$$

Let $n_k = \lambda_r^k$, $k_r < k < 2k_r$, and let

$$\sum_{|n|=n_k+1}^{n_{k+1}} |f_n(x)|^2 = \epsilon_k(x),\ k = k_r, k_r + 1, \ldots, 2k_r - 1;\ \sum_{k=k_r}^{2k_r-1} \epsilon_k(x) = D_r(x).$$

We set

$$\epsilon_k = \int_0^{2\pi} \epsilon_k(x)\, dx$$

and

$$D_r = \int_0^{2\pi} D_r(x)\, dx = \sum_{k=k_r}^{2k_r-1} \epsilon_k.$$

Since there are k_r terms ϵ_k, it follows that for at least one k, say $k(r)$,

$$k_r < k(r) < 2k_r - 1, \quad \epsilon_{k(r)} < D_r/k_r.$$

Thus $\epsilon_{k(r)}(x) < D_r/\alpha_r k_r$ for x outside a set E_r whose measure, $|E_r|$, does not exceed α_r. Since

$$\sum_{r=1}^{\infty} \alpha_r < \infty,$$

for almost every x

$$\epsilon_{k(r)}(x) < D_r/\alpha_r k_r$$

for all sufficiently large r. For such an x and all sufficiently large r with $k = k(r)$

$$(7) \qquad \sum_{|n|=n_k+1}^{n_{k+1}} |f_n(x)|^2 \log(|n|) < (\log \lambda_r^{2k_r})\epsilon_k(x) < \frac{2D_r \log \lambda_r}{\alpha_r}.$$

Since D_r does not exceed the left side of (6), the left side of (7) does not exceed 2^{-r}. Thus for almost every x

$$\sum_{\substack{k=k(r)\\ r>1}} \sum_{|n|=n_k+1}^{n_{k+1}} |f_n(x)|^2 \log(|n|) < \infty$$

since, except for a finite number of r values the terms satisfy (7) and so do not exceed 2^{-r}. This is sufficient to show as in our previous arguments (**1**, p. 396) that for almost every x, the $\{q_\nu\}$ partial sums of the series (5) converge to $f(x, y)$ for almost every y where the sequence $\{q_\nu\}$ takes on the values n, $n_{k(r)} < n < n_{k(r)+1}$, $r = 1, 2, \ldots$. The sequence $\{q_\nu\}$ is also of upper density one. This, together with (4), gives the second equality of part (i) of the theorem.

Our first step in proving the first equality of part (i) is to show that the sequences $\{p_\mu\}$ and $\{q_\nu\}$ already chosen in the proof of the second equality may be made the same. The sequence $\{p_\mu\}$ was chosen by the technique of Theorem 1 so as to insure the almost everywhere convergence of each sequence

$$\sum_{m=-p_\mu}^{p_\mu} c_{m,n} e^{imx}$$

to $f_n(x)$. The sequence $\{\lambda_r\}$ used in the proof of Theorem 1 may be taken to be the same as the sequence $\{\lambda_r\}$ already used in the proof of Theorem 2. Moreover the two $\{k_r\}$ sequences may be taken the same since, in each case, k_r was chosen large relative to a condition involving λ_r. By the remark following Theorem 1, for more than one-half the indices k, $k_r < k < 2k_r - 1$,

$$\epsilon_k^{(n)} < \frac{3(r+1)\, D_r(n)}{k_r}, \qquad n = 1, 2, \ldots, r.$$

As in our proof of the present theorem, for more than one-half the indices k,

$$\epsilon_k = \int_0^{2\pi} \epsilon_k(x)\, dx < 3\, D_r/k_r.$$

Hence, there is at least one index, say $k(r)$, for which both conditions are satisfied. Now the $\{p_\mu\}$ and the $\{q_\nu\}$ are chosen from the same blocks of integers, that is, n such that $\lambda_r^k < n < \lambda_r^{k+1}$, $k = k(r)$, $r = 1, 2, \ldots$.

It is easy to see that the same sequences $\{p_\mu\}$, $\{q_\nu\}$ can be chosen for any two functions of $L^2(\Omega)$, in particular for our $f(x, y)$ and for $g(x, y) = f(y, x)$. Since $s_{p,q}(x, y; g) = s_{q,p}(y, x; f)$ we may apply the second equality of part (i) to the sequence

$$s_{p_\mu, q_\nu}(x, y; g)$$

to obtain that almost everywhere

$$g(x, y) = \lim_{\nu\to\infty}\left[\lim_{\mu\to\infty} s_{p_\mu, q_\nu}(x, y; g)\right] = \lim_{\nu\to\infty}\left[\lim_{\mu\to\infty} s_{q_\nu, p_\mu}(y, x; f)\right] = f(y, x).$$

Since the $\{p_\mu\}$ take on the same values as the $\{q_\nu\}$ the first equality of part (i) of Theorem 2 is proved.

Now part (ii) follows easily. The difference

$$\sum_{n=-q_\nu}^{q_\nu} f_n(x)e^{iny} - f(x, y)$$

is smaller in absolute value than $1/s$ for (x, y) outside the set E_s when q_ν takes on values in the rth block, that is, $n_k < q_\nu < n_{k+1}$, $k = k(r)$. We may choose $r = r_s$ so large that

$$\sum_{s=1}^{\infty} |E_s| < \infty.$$

Also the difference

$$s_{p_\mu, q_\nu}(x, y; f) - \sum_{n=-q_\nu}^{q_\nu} f_n(x)e^{iny}$$

is smaller in absolute value than $1/s$ for (x, y) outside the set F_s, for q_ν in the rth block and p_μ in the Rth block, that is, $n_k < p < n_{k+1}$, $k = k(R)$. Again we may choose $R = R_s$ so large that

$$\sum_{s=1}^{\infty} |F_s| < \infty.$$

Hence, for (x, y) outside both E_s and F_s,

$$|s_{p_\mu, q_\nu}(x, y; f) - f(x, y)| < \frac{2}{s}, \quad \begin{matrix} n_k < p_\mu < n_{k+1},\ k = k(r_s) \\ n_k < q_\nu < n_{k+1},\ k = k(R_s). \end{matrix}$$

Almost every point (x, y) is outside all E_s and F_s for s sufficiently large. Now let P, the double sequence of positive integers, consist of the union of all P_s for all s where P_s is defined as all (p, q) such that p belongs to the R_sth block and q to the r_sth block. For the double sequence P

$$\frac{\sigma(n_{k(R)+1}, n_{k(r)+1})}{n_{k(R)+1}\, n_{k(r)+1}} > \left(1 - \frac{1}{\lambda_R}\right)\left(1 - \frac{1}{\lambda_r}\right), \qquad r = r_s, R = R_s$$

which approaches one as s increases to ∞ so that P is of upper density one. Part (ii) follows from this.

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ON INTEGRATION OF VECTOR-VALUED FUNCTIONS

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1. Introduction. Among the variety of integrals which have been devised for integrating vector-valued functions the most widely used is that of Bochner (**2**), perhaps because of the simplicity of its formulation. Other approaches, including one by Birkhoff (**1**), have yielded more general integrals yet none of them seems to have supplanted the Bochner integral to a significant extent.

Another simple approach is that of Graves (**4**). This is an adaptation of the Riemann definition, and the resulting integral has most of the properties of the ordinary Riemann integral. A noteworthy exception is that there exist functions which are everywhere discontinuous and yet are Graves integrable. In sharp contrast to the real variable case the Bochner (Lebesgue) integral does not include the Graves (Riemann) integral. Neither does the Graves integral include that of Bochner.

In the present paper we show that the Graves integral can be generalized in a simple way to produce an integral which includes the Bochner integral as a special case, and is equivalent to the Birkhoff integral for functions defined on a bounded Lebesgue measurable set in n-dimensional Euclidean space. This generalization stems from the fact that the Lebesgue measurability of a finite real-valued function f, on a measurable set E, is equivalent to the validity of the well-known Lusin condition (**9**, p. 72) for f. It has been pointed out by Hildebrandt (**6**) that a definition of the Lebesgue integral due to Hahn (**5**) is based on the Lusin property and that this suggests an alternate approach to the Bochner integral. Bourbaki (**3**, p. 180) gives a definition of measurability for a function f, defined on a locally compact space E with values in an arbitrary topological space F, which is also based on the Lusin condition in that f is required to be continuous on each of a collection of compact sets with total measure approximating that of E. It turns out that when the range space is a Banach space this definition is equivalent to Bochner measurability (**3**, Theorem 3, p. 189). We notice, however, that there exist fairly simple functions which are Graves (Riemann) integrable but not measurable in the Bochner or Bourbaki senses, nor in any sense that implies the Lusin property. The classical example is that of Graves (**4**, p. 166) which involves the space M of bounded real functions $f(t)$ on $0 \leqslant t \leqslant 1$, with

$$\|f(t)\| = \sup_{0 \leqslant t \leqslant 1} |f(t)|.$$

Let $x(\alpha) = f_\alpha(t)$ where $f_\alpha(t) = 0$ on $0 \leqslant t \leqslant \alpha$, and $f_\alpha(t) = 1$ on $\alpha < t \leqslant 1$. Thus $x(\alpha)$ is defined on $0 \leqslant \alpha \leqslant 1$ and is everywhere discontinuous there.

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On the other hand, this function is integrable in the sense of Birkhoff and of Jeffery **(7)**. These facts suggest a weakening of the Lusin condition in which we replace the sets on which the function is required to be continuous **(5; 6; 3)** by sets over which the function is integrable in the Graves sense (generalized in a natural way so as to be defined on closed sets), and hence may in some cases be everywhere discontinuous on these sets. A definition of measurability is based on this weakened condition and the Hahn approach is then used in defining our generalized integral.

2. Notation. Throughout this paper X will denote an arbitrary linear normed complete space, or Banach space, R the space of real numbers, $x(\alpha)$, $y(\alpha)$ functions valued in X, and $f(\alpha)$, $g(\alpha)$ real-valued functions. The symbol $[a, b]$ will denote a closed interval on the real line, P, P', F closed subsets of $[a, b]$, and $|E|$ the Lebesgue measure of a measurable set E.

3. A Graves integral defined over a closed set.

Definition 3.1. Let $x(\alpha)$ be defined and bounded on P. Let π be a subdivision of $[a, b]$ into subintervals (α_{i-1}, α_i); let $\Delta\alpha_1$ denote the closed interval $[\alpha_0, \alpha_1]$ and $\Delta\alpha_i$, $i>1$, denote the half-open interval $(\alpha_{i-1}, \alpha_i]$. Let $N\pi$ be the maximum of the differences $\alpha_i - \alpha_{i-1}$, called the *norm* of π. If X contains an element L such that for every $\epsilon > 0$ there exists $\delta > 0$ with

$$\left\| \sum_{i=1}^{n} x(\xi_i)|P \cap \Delta\alpha_i| - L \right\| < \epsilon$$

for every subdivision with $N\pi < \delta$, and every choice of ξ_i in $P \cap \Delta\alpha_i (i = 1, 2, \ldots, n)$ then L is the Graves integral, or G-integral, of $x(\alpha)$ over P and we write

$$(G) \int_P x(\alpha)\, d\alpha = L.$$

It is not difficult to see that when P is a closed interval the G-integral reduces to the original Graves integral.

Because of the frequency and importance of its uses in the remainder of this paper we state the following rssult, which has been proved in a variety of ways by several writers including Birkhoff **(1)**, Jeffery **(7)**, and Macphail **(8)**.

THEOREM 3.1. *Let $e_1, e_2, \ldots, e_n$ be any n disjoint Lebesgue measurable sets on a measurable set E, $|E| < \infty$, ξ_i any point on e_i, and $S = \sum x(\xi_i)|e_i|$ where x is a bounded function on E with values in a space X. Let*

$$e_{i1}, e_{i2}, \ldots, e_{ik_i}$$

be a subdivision of e_i into disjoint measurable sets, and ξ_{ij}, ξ'_{ij} any points on e_{ij}. Then

$$\left|\left|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\{x(\xi_{ij})-x(\xi'_{ij})\}|e_{ij}|\right|\right| < \sup\left|\left|\sum_{i=1}^{n}\{x(\xi_i)-x(\xi'_i)\}|e_i|\right|\right|$$

and

$$\left|\left|\sum_{i=1}^{n}\sum_{j=1}^{k_i}x(\xi_{ij})|e_{ij}|-S\right|\right| < \sup\left|\left|\sum_{i=1}^{n}\{x(\xi_i)-x(\xi'_i)\}|e_i|\right|\right|.$$

One consequence of this result is the following theorem.

THEOREM 3.2. *Let $x(\alpha)$ be defined and bounded on P. A necessary and sufficient condition for the existence of the* G-*integral of $x(\alpha)$ over P is that there exist a sequence of subdivisions π_n of $[a, b]$ such that*

$$\lim_{n\to\infty}\sum_{\pi_n}x(\xi_{ni})|P\cap\Delta\alpha_{ni}|$$

exists, ξ_{ni} any point on $P \cap \Delta\alpha_{ni}$.

Proof. The necessity is obvious in view of definition 3.1. To prove the sufficiency choose any $\epsilon > 0$ and consider a sequence $\{\pi_n\}$ which yields a limit J. Then there exists an n_0 such that for $n > n_0$ we have $||\Sigma_{\pi_n} - J|| < \frac{1}{2}\epsilon$. If $n > n_0$ is fixed and ξ_{ni}, ξ'_{ni} are allowed to be any points in $P \cap \Delta\alpha_{ni}$ it follows that

$$\left|\left|\sum\{x(\xi_{ni})-x(\xi'_{ni})\}|P\cap\Delta\alpha_{ni}|\right|\right| < \epsilon.$$

Then let π_k be any subdivision of $[a, b]$, not necessarily in the sequence, with $N\pi_k$ sufficiently small to insure that the total length of the intervals of π_k which contain points of subdivision of π_n is less than ϵ/M, where $M = \sup ||x(\alpha)||$, α on P. Suitable applications of Theorem 3.1 show that $||\Sigma_{\pi_k} - J||$ is less than a fixed multiple of ϵ.

In the next theorem we list several properties of the G-integral which we shall use in making our generalization. The proofs follow from the definition and Theorem 3.1 by standard arguments and are omitted.

THEOREM 3.3. (a) *If x, y, and f are* G-*integrable over P, and $||x(\alpha)|| < f(\alpha)$ for α on P, then*

(i) $$(G)\int_P(x+y)d\alpha = (G)\int_P x\,d\alpha + (G)\int_P y\,d\alpha,$$

(ii) $$\left|\left|(G)\int_P x\,d\alpha\right|\right| < (G)\int_P f\,d\alpha.$$

(b) *If $P \cap P' = 0$ and $x(\alpha)$ is* G-*integrable over the sets P and P' then it is integrable over $P \cup P'$ and*

$$(G)\int_{P\cup P'}x\,d\alpha = (G)\int_P x\,d\alpha + (G)\int_{P'}x\,d\alpha.$$

(c) *If P' is contained in P then $x(\alpha)$ is* G-*integrable over P' if it is* G-*integrable over P and*

$$\left|\left|(G)\int_P x d\alpha - (G)\int_{P'} x d\alpha\right|\right| < M|P - P'|,\ M = \sup_{\alpha\epsilon P}||x(\alpha)||.$$

(d) *If $x_n(\alpha)$ $(n = 1, 2, \ldots)$ is* G-*integrable over P and if $x_n(\alpha)$ converges uniformly to $x(\alpha)$ on P then $x(\alpha)$ is* G-*integrable over P and*

$$(G)\int_P x_n(\alpha)d\alpha \to (G)\int_P x(\alpha)d\alpha.$$

4. The generalized Graves integral. In this and the remaining sections E, E_i will denote bounded Lebesgue measurable sets of the real line.

Definition 4.1. A function $x(\alpha)$ defined on a set E with values in X is P_ϵ-*measurable* on E if for every $\epsilon > 0$ there exists a closed set P, contained in E, such that (i) $|E - P| < \epsilon$, and (ii) $x(\alpha)$ is G-integrable over P.

This is a generalization of the classical Lusin condition, to which it is equivalent when $X = R$. For arbitrary X we observe that if $x(\alpha)$ is continuous on P contained in E, with $|E - P| < \epsilon$, it is Graves integrable on P and therefore P_ϵ-measurable. Conversely, if a real-valued function $x(\alpha)$ is P_ϵ-measurable on E and hence G-integrable on $P' \supset E$, $|E - P'| < \frac{1}{2}\epsilon$, a standard argument shows that the measure of the set of its discontinuities on P' is zero. Then there exists a set P in E, on which $x(\alpha)$ is continuous with $|E - P| < \epsilon$.

On the other hand, the function cited in the introduction is P_ϵ-measurable without satisfying the Lusin condition.

Definition 4.2. If $x(\alpha)$ is P_ϵ-measurable on E and if there is an element I in X such that given $\eta > 0$ there exists $\epsilon > 0$ with

$$\left|\left|(G)\int_P x(\alpha)d\alpha - I\right|\right| < \eta$$

for every P contained in E, $|E - P| < \epsilon$, over which $x(\alpha)$ is G-integrable, then we say that I is the G*-*integral* of $x(\alpha)$ over E and denote it by (G^*) $\int_E x(\alpha)d\alpha$.

THEOREM 4.1. *If $x(\alpha)$ is P_ϵ-measurable on E a necessary and sufficient condition for*

$$(G^*)\int_E x(\alpha)d\alpha$$

to exist is that for every $\eta > 0$ there exist $\delta > 0$ such that if P and P' are two closed sets in E, with measures greater than $|E| - \delta$, for which

$$(G)\int_P x d\alpha,\ (G)\int_{P'} x d\alpha$$

exist then

$$\left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G})\int_{P'} x d\alpha\right|\right| < \eta. \tag{4.1}$$

The proof is easily obtained by a standard argument and will be omitted.

THEOREM 4.2. *If $x(\alpha)$ is* G*-*integrable over E it is* G*-*integrable over every measurable subset e contained in E.*

Proof. Given $\epsilon > 0$, let P be a closed set with $|E - P| < \frac{1}{2}\epsilon$ and such that $x(\alpha)$ is G-integrable over P. Let e be any measurable subset of E. Then eP is measurable and so it contains a closed set P' with $|eP - P'| < \frac{1}{2}\epsilon$. Moreover

$$|e - eP| < |E - P| < \tfrac{1}{2}\epsilon.$$

Then $|e - P'| < \epsilon$. Since $x(\alpha)$ is G-integrable over P' by Theorem 3.3(c) our conclusion follows.

We next prove the analogue of the fact, basic in real variable theory, that every bounded Lebesgue measurable function on a set E is Lebesgue integrable.

THEOREM 4.3. *If $x(\alpha)$ is bounded and P_ϵ-measurable on E it is* G*-*integrable over E.*

Proof. Let $M = \sup||x(\alpha)||$ for α in E. Let $\eta > 0$ be given and let P and P' be closed sets, contained in E, on which $x(\alpha)$ is G-integrable and such that the measure of each set is greater than $|E| - (\eta/2M)$. Then $x(\alpha)$ is G-integrable over $P \cap P'$ by Theorem 3.3(c). Moreover

$$|P' - (P \cap P')| < \frac{\eta}{2M}, \ |P - (P \cap P')| < \frac{\eta}{2M}.$$

Hence, by the second part of Theorem 3.3(c),

$$\left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G})\int_{P'} x d\alpha\right|\right|$$
$$< \left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G})\int_{P\cap P'} x d\alpha\right|\right| + \left|\left|(\mathrm{G})\int_{P\cap P'} x d\alpha - (\mathrm{G})\int_{P'} x d\alpha\right|\right|$$
$$< M\cdot\frac{\eta}{2M} + M\cdot\frac{\eta}{2M} = \eta.$$

THEOREM 4.4. *If $E_1 \cap E_2 = 0$, and if $x(\alpha)$ is* G*-*integrable over E_1 and E_2 then it is* G*-*integrable over $E_1 \cup E_2$ and*

$$(\mathrm{G}^*)\int_{E_1\cup E_2} x d\alpha = (\mathrm{G}^*)\int_{E_1} x d\alpha + (\mathrm{G}^*)\int_{E_2} x d\alpha. \tag{4.2}$$

Proof. Let $\{P_n\}$, $\{P_n'\}$ be sequences of sets over each of which $x(\alpha)$ is G-integrable and with $P_n \subset E_1$, $P_n' \subset E_2$, for all n. Suppose that

$$|P_n| \to |E_1|, \ |P_n'| \to |E_2| \qquad n \to \infty.$$

Then $|P_n + P_n'| \to |E_1 + E_2|$. For each n, by Theorem 3.3(b),

$$(\mathrm{G})\int_{P_n} x\,d\alpha + (\mathrm{G})\int_{P_n'} x\,d\alpha = (\mathrm{G})\int_{P_n \cup P_n'} x\,d\alpha.$$

If $x(\alpha)$ is G*-integrable on $E_1 \cup E_2$ it follows that (4.2) holds.

It remains to be shown that $x(\alpha)$ is G*-integrable on $E_1 \cup E_2$. Let I denote the right side of (4.2). Given $\eta > 0$ there exists $\epsilon > 0$ such that if P and P' are closed sets contained in E_1, E_2 respectively, with $|E_1 - P| < \epsilon$, $|E_2 - P'| < \epsilon$, and if $x(\alpha)$ is G-integrable on P and P', we have

$$\left|\left|(\mathrm{G})\int_P x\,d\alpha - (\mathrm{G}^*)\int_{E_1} x\,d\alpha\right|\right| < \tfrac{1}{3}\eta, \quad \left|\left|(\mathrm{G})\int_{P'} x\,d\alpha - (\mathrm{G}^*)\int_{E_2} x\,d\alpha\right|\right| < \tfrac{1}{3}\eta.$$

Now, suppose F is any closed set contained in $E_1 \cup E_2$, with $|(E_1 \cup E_2) - F| < \frac{1}{2}\epsilon$, and on which $x(\alpha)$ is G-integrable. It follows that

$$|E_1 - (F \cap E_1)| = |E_1 - F| < |(E_1 \cup E_2) - F| < \tfrac{1}{2}\epsilon.$$

Similarly $|E_2 - (F \cap E_2)| < \frac{1}{2}\epsilon$. Then let P, P' contained in $F \cap E_1$, $F \cap E_2$ respectively, be such that

$$|E_1 - P| < \epsilon, \ |E_2 - P'| < \epsilon, \ |F - (P \cup P')| < \frac{\eta}{3M},$$

where $M = \sup||x(\alpha)||$ on F. $x(\alpha)$ is G-integrable on the sets P and P' by Theorem 3.3(c). Hence

$$\begin{aligned}
&\left|\left|(\mathrm{G})\int_F x\,d\alpha - I\right|\right| \\
&< \left|\left|(\mathrm{G})\int_F x\,d\alpha - (\mathrm{G})\int_{P \cup P'} x\,d\alpha\right|\right| + \left|\left|(\mathrm{G})\int_P x\,d\alpha - (\mathrm{G}^*)\int_{E_1} x\,d\alpha\right|\right| \\
&\qquad + \left|\left|(\mathrm{G})\int_{P'} x\,d\alpha - (\mathrm{G}^*)\int_{E_2} x\,d\alpha\right|\right| \\
&< \eta.
\end{aligned}$$

Definition 4.3. If a set function ν is defined on the class of Lebesgue measurable subsets of E with values in X, and if for every $\epsilon > 0$ there exists $\delta > 0$ such that $||\nu(e)|| < \epsilon$ for every subset e of E with $|e| < \delta$, then we say that ν is ***absolutely continuous*** over the measurable subsets of E.

THEOREM 4.5. *Suppose $x(\alpha)$ is* G*-*integrable over a set E. Then the* G*-*integral is an absolutely continuous function of the measurable sets e contained in E.*

Proof. Let $\eta > 0$ be given. Fix $\delta > 0$ so that condition (4.1) of Theorem 4.1 holds. Now consider any set e contained in E with $|e| < \delta$. Let P be any closed set contained in e such that

$$\left|\left|(\mathrm{G}^*)\int_e x\,d\alpha - (\mathrm{G})\int_P x\,d\alpha\right|\right| < \eta.$$

Also let F be a closed set contained in E-e, with $|E - F| < \epsilon$, and on which $x(\alpha)$ is G-integrable. Then

$$\left\|(G)\int_P x\,d\alpha\right\| = \left\|(G)\int_{P^*P} x\,d\alpha - (G)\int_P x\,d\alpha\right\| < \eta$$

and

$$\left\|(G^*)\int_e x\,d\alpha\right\| < \left\|(G^*)\int_e x\,d\alpha - (G)\int_P x\,d\alpha\right\| + \left\|(G)\int_P x\,d\alpha\right\| < 2\eta.$$

Since η is arbitrary our conclusion follows.

THEOREM 4.6. *If $x(\alpha)$ is G^*-integrable over E, and e represents a measurable subset of E, then*

$$(G^*)\int_e x(\alpha)\,d\alpha$$

is a completely additive set function over E.

Proof. Let $e_1, e_2, \ldots$ be a sequence of disjoint measurable sets on E and let $e_1 + \ldots + e_n = E_n$ and

$$\sum_{i=1}^{\infty} e_i = e.$$

Then $x(\alpha)$ is G-integrable over e and E_n by Theorem 4.2, and by Theorem 4.4 we have

$$\sum_{i=1}^{n} (G^*)\int_{e_i} x\,d\alpha = (G^*)\int_{E_n} x\,d\alpha$$

and

$$(G^*)\int_{E_n} x\,d\alpha + (G^*)\int_{e-E_n} x\,d\alpha = (G^*)\int_e x\,d\alpha.$$

Now $|e - E_n| \to 0$ as $n \to \infty$. Hence, by Theorem 4.5,

$$\sum_{i=1}^{\infty} (G)\int_{e_i} x\,d\alpha = \lim_{n\to\infty} (G^*)\int_{E_n} x\,d\alpha = (G^*)\int_e x\,d\alpha.$$

Although we have restricted ourselves to sets on the real line for the sake of simplicity in writing, it is clear that the above definitions and theorems can be extended to a function with its values in X and defined on any bounded Lebesgue measurable set in an n-dimensional Euclidean space. In addition, the usual procedure of taking limits would lead to a definition of the G^*-integral in cases where E is not bounded and $|E|$ is not finite.

5. Sequences of P_ϵ-measurable functions. We now consider two important properties of sequences of P_ϵ-measurable functions which generalize corresponding results in the Lebesgue theory.

Definition 5.1. A sequence of functions $\{x_n(\alpha)\}$ defined on E converges to $x(\alpha)$ in E *almost uniformly* if, given $\epsilon > 0$, there exists a set E' contained in E such that $|E - E'| < \epsilon$ and $x_n(\alpha)$ converges uniformly to $x(\alpha)$ in E'.

LEMMA 5.1. *Let $\{x_n(\alpha)\}$ be a sequence of P_ϵ-measurable functions defined on E with values in X. If $x_n(\alpha)$ converges to $x(\alpha)$ in E almost uniformly then $x(\alpha)$ is P_ϵ-measurable on E.*

Proof. Suppose $\epsilon > 0$ is given. Then there exists a set E', contained in E, on which $x_n(\alpha)$ converges to $x(\alpha)$ uniformly and such that $|E - E'| < \epsilon/2$. Now for each n there exists a closed set P_n contained in E with $|E - P_n| < \epsilon/2^{n+2}$ and such that $x_n(\alpha)$ is G-integrable over P_n. Furthermore, there exists a closed set P_n' in the measurable set E' such that $|E' - P_n'| < \epsilon/2^{n+2}$. Hence we may set $F_n = P_n \cap P_n'$ for each n and $x_n(\alpha)$ will be G-integrable on F_n, by Theorem 3.3(c), with $|E' - F_n| < \epsilon/2^{n+1}$. Then the intersection of the sequence of sets $\{F_n\}$ is a closed set F such that $|E - F| < \epsilon$ and each $x_n(\alpha)$ is G-integrable on F. Hence (G) $\int_F x(\alpha)\, d\alpha$ exists by Theorem 3.3(d). Since ϵ is arbitrary it follows that $x(\alpha)$ is P_ϵ-measurable on E.

LEMMA 5.2. *If $x(\alpha)$ is G^*-integrable over a set E and if $f(\alpha)$ is a real-valued summable function over E, with $||x(\alpha)|| < f(\alpha)$ for every α in E, then*

$$\left|\left|(\mathrm{G}^*)\int_E x(\alpha)d\alpha\right|\right| < (\mathrm{G}^*)\int_E f(\alpha)d\alpha.$$

Proof. This follows immediately from the second part of Theorem 3.3(*a*) and the definition of the G*-integral.

THEOREM 5.1. *Let $\{x_n(\alpha)\}$ be a sequence of P_ϵ-measurable functions defined on E with values in X. Suppose $||x_n(\alpha)|| < f(\alpha)$ for all values of n, and all α in E, where $f(\alpha)$ is a real-valued summable function over E. Suppose also that $x_n(\alpha)$ converges to $x(\alpha)$ in E almost uniformly. Then $x_n(\alpha)$ is* G**-integrable over E for each n, $x(\alpha)$ is* G**-integrable over E, and*

$$\lim_{n\to\infty} (\mathrm{G}^*)\int_E x_n(\alpha)d\alpha = (\mathrm{G}^*)\int_E x(\alpha)d\alpha. \tag{5.1}$$

Proof. $x(\alpha)$ is P_ϵ-measurable on E by Lemma 5.1. Furthermore, given $\eta > 0$ there exists $\delta > 0$ such that for e contained in E and $|e| < \delta$,

$$(\mathrm{G}^*)\int_e f(\alpha)d\alpha < \tfrac{1}{4}\eta.$$

Let P and P' be any two closed sets in E, on each of which $x(\alpha)$ is G-integrable, and such that $|E - P| < \delta$, $|E - P'| < \delta$. Then

$$\begin{aligned}
&\left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G})\int_{P'} x d\alpha\right|\right| \\
&\quad= \left|\left|(\mathrm{G}^*)\int_P x d\alpha - (\mathrm{G}^*)\int_{P'} x d\alpha\right|\right| \\
&\quad= \left|\left|(\mathrm{G}^*)\int_{P-P'} x d\alpha - (\mathrm{G}^*)\int_{P'-P} x d\alpha\right|\right| \\
&\quad< \left|\left|(\mathrm{G}^*)\int_{P-P'} x d\alpha\right|\right| + \left|\left|(\mathrm{G}^*)\int_{P'-P} x d\alpha\right|\right| < \eta.
\end{aligned}$$

Hence $x(\alpha)$ is G*-integrable over E by Theorem 4.1, and the same argument shows that for each n, $x_n(\alpha)$ is G*-integrable over E.

Finally, let $x(\alpha) = x_n(\alpha) - y_n(\alpha)$. By our hypothesis there exists a set E' contained in E, such that $|E - E'| < \delta$, and a fixed positive integer N such that for $n > N$ we have

$$||x(\alpha) - x_n(\alpha)|| < \frac{\eta}{2|E|}$$

on E'. Then

$$\begin{aligned}
&\left|\left|(\mathrm{G}^*) \int_E \{x(\alpha) - x_n(\alpha)\} d\alpha\right|\right| \\
&\quad = \left|\left|(\mathrm{G}^*) \int_{E'} y_n(\alpha) d\alpha + (\mathrm{G}^*) \int_{E-E'} y_n(\alpha) d\alpha\right|\right| \\
&\quad < \left|\left|(\mathrm{G}^*) \int_{E'} y_n(\alpha) d\alpha\right|\right| + \left|\left|(\mathrm{G}^*) \int_{E-E'} y_n(\alpha) d\alpha\right|\right| \\
&\quad < \left|\left|(\mathrm{G}^*) \int_{E'} y_n(\alpha) d\alpha\right|\right| + (\mathrm{G}^*) \int_{E-E'} 2f(\alpha) d\alpha \\
&\quad < \frac{\eta}{2|E|} \cdot |E'| + \tfrac{1}{2}\eta < \eta .
\end{aligned}$$

This completes the proof.

COROLLARY. *If* $\{x_n(\alpha)\}$ *is uniformly bounded on* E, *and* $x_n(\alpha)$ *converges to* $x(\alpha)$ *in* E *almost uniformly, then* (5.1) *holds.*

For a sequence of real-valued Lebesgue measurable functions on a bounded set E, convergence almost everywhere is equivalent to convergence almost uniformly. It is clear then that Theorem 5.1 is a generalization of the well-known dominated convergence theorem of Lebesgue. However, because of the failure of a theorem of the Egoroff type to hold, in general, for P_e-measurable functions the traditional hypothesis of the Lebesgue theorem cannot be retained, and we must require specifically that the functions converge almost uniformly.

6. The equivalence of the G*-integral and the Birkhoff integral. It is easy to show directly that the G*-integral includes the Bochner integral. However, the Birkhoff integral also includes that of Bochner and is more general than the latter. For this reason we shall compare the G*-integral with that of Birkhoff.

Macphail **(8)** points out that if $x(\alpha)$ is bounded on E the infinite partitions of Birkhoff may be replaced by finite partitions. If $\mathfrak{P}$ is a finite partition of E into measurable subsets e_i, and if we define

$$S(\mathfrak{P}) = \sum_i x(\xi_i)|e_i|, \; D(\mathfrak{P}) = \sum_i [x(\xi_i') - x(\xi_i'')]|e_i|,$$

and $\omega(\mathfrak{P}) = \sup||D(\mathfrak{P})||$, where ξ_i, ξ_i', ξ_i'' are arbitrary points in e_i the diameter of the integral range which appears in Birkhoff's definition **(1**, p. 367**)**

is precisely $\omega(\mathfrak{P})$. Then a bounded function $x(\alpha)$ is Birkhoff integrable if and only if there exists (**1**, Theorem 13) a sequence of partitions $\{\mathfrak{P}_n\}$ such that $\omega(\mathfrak{P}_n) \to 0$. The following lemmas are consequences of the above definitions, with the aid of Theorem 3.1.

LEMMA 6.1. *If $\mathfrak{P}$ is any finite partition of E on which a bounded function $x(\alpha)$ is (Bk)-integrable then*

$$\left|\left|(Bk)\int_E x d\alpha - S(\mathfrak{P})\right|\right| < \omega(\mathfrak{P}).$$

LEMMA 6.2. *If Q is a set consisting of a selection $\mathfrak{Q}$ of the subsets comprising $\mathfrak{P}$ then*

$$\left|\left|(Bk)\int_Q x d\alpha - S(\mathfrak{Q})\right|\right| < \omega(\mathfrak{Q}) < \omega(\mathfrak{P}).$$

In (**1**) the function $x(\alpha)$ is considered to be defined on an abstract domain on which a measure is defined. However, as in the previous sections, we restrict the present discussion to a function defined on a bounded Lebesgue measurable linear set, observing that the same proofs hold for n-dimensional sets. The Birkhoff integral of $x(\alpha)$ over E will be denoted by $(Bk)\int_E x(\alpha)d\alpha$.

THEOREM 6.1. *If $x(\alpha)$ is Birkhoff integrable over a set E then it is also* G*-*integrable over E to the same value.*

Proof. Suppose first that $x(\alpha)$ is bounded on E, and let $M > 0$ be such that $||x(\alpha)|| < M$ for all α in E. Let $\{\mathfrak{P}_n\}$ be a sequence of finite partitions of the set E which yield a Birkhoff integral. That is, for each n, $E = e_{n1} + e_{n2} + \ldots + e_{nk}$. Then, given $\epsilon > 0$ we can choose closed sets e^c_{ni} contained in the e_{ni} such that the measure of $E^c_n = e^c_{n1} + e^c_{n2} + \ldots + e^c_{nk}$ differs from the measure of E by less than $\epsilon/2^n$.

Let $F = E^c_1 \cap E^c_2 \cap \ldots$. This is a closed set and its measure differs from that of E by less than ϵ. We show that

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} x(\xi_i)|F \cap e^c_{ni}| = (Bk)\int_F x(\alpha)d\alpha.$$

First of all (**1**, Theorem 14), $x(\alpha)$ is (Bk)-integrable on F. Also, the sets $F \cap e^c_{ni}$ form a partition of F which we may denote by $\mathfrak{P}_{n(F)}$. Clearly $\mathfrak{P}_{n(F)}$ consists of a selection of sets from $\mathfrak{P}_n'$, the partition of E formed by the sets

$$F \cap e^c_{ni},\ e^c_{ni} - F, \text{ and } e_{ni} - e^c_{ni}, \qquad i = 1, 2, \ldots, k_n.$$

Moreover, $\mathfrak{P}_n'$ is a refinement of the partition $\mathfrak{P}_n$ of E. Then, by Lemma 6.2,

$$\begin{aligned}\left|\left|(Bk)\int_F x d\alpha - \sum_{i=1}^{k_n} x(\xi_i)|F \cap e^c_{ni}|\right|\right| \\ = \left|\left|(Bk)\int_F x d\alpha - S(\mathfrak{P}_{n(F)})\right|\right| \\ < \omega(\mathfrak{P}_{n(F)}) < \omega(\mathfrak{P}_n'),\end{aligned}$$

and by Theorem 3.1, $\omega(\mathfrak{P}_n') < \omega(\mathfrak{P}_n)$, where $\omega(\mathfrak{P}_n)$ is associated with the original sequence $\{\mathfrak{P}_n\}$ and approaches zero as $n \to \infty$. This leads to the desired conclusion. Then, given $\eta > 0$ we can find and fix an m, depending on η, such that

$$\left\|\sum_{i=1}^{k_m} x(\xi_i)|F \cap e^c_{mi}| - (Bk)\int_F x d\alpha\right\| < \omega(\mathfrak{P}_m) < \eta.$$

Now the closed sets e^c_{mi} are disjoint and finite in number. Let d be the minimum of the distances between any two of these closed sets. Then, taking an interval $[a, b]$ containing E divide it into subintervals of length less than d, that is,

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_n = b.$$

Consider sets of the form $F \cap \Delta\alpha_i$ where the $\Delta\alpha_i$ are as described in definition 3.1. We see that each set $F \cap \Delta\alpha_i (i = 1, 2, \ldots, n)$ is equivalent to a set $F \cap e^c_{mj} \cap \Delta\alpha_i$ for some j, and the collection of such sets forms a refinement of the partition $\mathfrak{P}_{m(F)}$. Then it follows that

$$\begin{aligned}&\left\|\sum_{i=1}^{n} x(\xi_i)|F \cap \Delta\alpha_i| - (Bk)\int_F x(\alpha)d\alpha\right\| \\ &\quad < \left\|\sum_{i=1}^{n} x(\xi_i)|F \cap \Delta\alpha_i| - \sum_{i=1}^{k_m} x(\xi_i)|F \cap e^c_{mi}|\right\| \\ &\qquad + \left\|\sum_{i=1}^{k_m} x(\xi_i)|F \cap e^c_{mi}| - (Bk)\int_F x(\alpha)d\alpha\right\| \\ &\quad < 2\omega(\mathfrak{P}_m) < 2\eta.\end{aligned}$$

Thus $x(\alpha)$ is G-integrable on the closed set F where $|E - F| < \epsilon$. Hence $x(\alpha)$ is P_ϵ-measurable over E and being bounded it is G*-integrable over E by Theorem 4.3. Further, the G-integral equals the Birkhoff integral on F. Hence, as $|E - F| \to 0$, we have (G) $\int_F x d\alpha$ approaching the limit (Bk) $\int_E x d\alpha$ since the Birkhoff integral is absolutely continuous and a completely additive set function on E. Therefore,

$$(G^*)\int_E x d\alpha = (Bk)\int_E x d\alpha.$$

If $x(\alpha)$ is unbounded on E but is Birkhoff integrable there it is also Birkhoff integrable on every measurable set e contained **(1)** in E and it follows that as $|e| \to |E|$,

$$(Bk)\int_e x d\alpha \to (Bk)\int_E x d\alpha.$$

Now, given $\epsilon > 0$, if E' is a measurable subset of E over which $x(\alpha)$ is bounded and such that $|E - E'| < \frac{1}{2}\epsilon$ then $(Bk)\int_{E'} x d\alpha$ exists and hence $(G^*)\int_{E'} x d\alpha$ exists. Then there is a closed set P contained in E' with $|E' - P| < \frac{1}{2}\epsilon$ and such that (G) $\int_P x d\alpha$ exists. Hence $x(\alpha)$ is P_ϵ-measurable on E. Then, given

$\eta > 0$ there exists $\delta > 0$ such that for P, P' contained in E, with $|E - P| < \delta$, $|E - P'| < \delta$, and on each of which $x(\alpha)$ is G-integrable,

$$\begin{aligned}\left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G})\int_{P'} x d\alpha\right|\right| &= \left|\left|(Bk)\int_P x d\alpha - (Bk)\int_{P'} x d\alpha\right|\right| \\ &< \left|\left|(Bk)\int_{P-P'} x d\alpha\right|\right| + \left|\left|(Bk)\int_{P'-P} x d\alpha\right|\right| < \eta.\end{aligned}$$

Thus $x(\alpha)$ is G*-integrable on E.

Finally, since $x(\alpha)$ is P_ϵ-measurable on E there is a sequence of closed sets $\{P_n\}$ with $|P_n| \to |E|$ and on each of which $x(\alpha)$ is G-integrable. Hence

$$(Bk)\int_E x d\alpha = \lim_{|P_n| \to |E|} (Bk)\int_{P_n} x d\alpha = \lim_{|P_n| \to |E|} (\mathrm{G})\int_{P_n} x d\alpha = (\mathrm{G}^*)\int_E x d\alpha.$$

COROLLARY. *If $x(\alpha)$ is Bochner integrable over E then it is also G*-integrable over E to the same value.*

This follows at once from our theorem and the proof that Birkhoff's integral includes that of Bochner (**1**, p. 377).

The fact that the everywhere discontinuous function given in the introduction is G*-integrable, being P_ϵ-measurable and bounded, but is not Bochner (strongly) measurable, shows that the converse does not hold.

THEOREM 6.2. *If $x(\alpha)$ is G*-integrable over E then it is also Birkhoff integrable over E to the same value.*

Proof. First suppose that $x(\alpha)$ is bounded on E, $||x(\alpha)|| < M$ for all α in E. Given $\epsilon > 0$ there exists a closed set P, with

$$|E - P| < \frac{\epsilon}{4M},$$

on which $x(\alpha)$ is G-integrable. By a suitable subdivision π we can partition this set into sets $e_i = P \cap \Delta\alpha_i$ and have

$$\left|\left|\sum_{\pi} [x(\xi_i') - x(\xi_i'')]|e_i|\right|\right| < \frac{\epsilon}{2},$$

for all ξ_i', ξ_i'' in e_i. Now by taking a set J, the complement of P in E, plus the sets e_i, we have a partition $\mathfrak{P}$ of the whole set E and can form the sum

$$D(\mathfrak{P}) = \sum_{\pi} [x(\xi_i') - x(\xi_i'')]|e_i| + \{[x(\xi_1) - x(\xi_2)]|J|, \xi_1, \xi_2 \text{ in } J\}.$$

Then

$$\sup\left|\left|D(\mathfrak{P})\right|\right| < \frac{\epsilon}{2} + \frac{\epsilon}{4M}\cdot 2M = \epsilon.$$

By starting with a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$ we can construct a sequence of partitions $\mathfrak{P}_n$ such that

$$\omega(\mathfrak{P}_n) = \sup \|D(\mathfrak{P}_n)\| \to 0.$$

Thus the Birkhoff integral over E exists and is clearly equal to $(G^*) \int_E x d\alpha$.

Next, suppose that $x(\alpha)$ is unbounded on E. Then, given $\epsilon > 0$, we have to find a partition under which $\sum x(\xi_i)|e_i|$ is unconditionally summable and the diameter of the integral range is less than ϵ.

Let $E_1, E_2, \ldots$ be non-overlapping sets in E such that $\sum |E_j| = |E|$ and $x(\alpha)$ is bounded on each E_j. For each E_j let $\mathfrak{E}_j$ be a partition into sets e_{ji} such that $\omega(\mathfrak{E}_j) < \epsilon_j$, $\sum \epsilon_j = \frac{1}{2}\epsilon$. Let $\delta > 0$ be such that for a measurable set e with $|e| < \delta$ we have

$$\left\| (G^*) \int_e x(\alpha) d\alpha \right\| < \tfrac{1}{2}\epsilon.$$

Choose N such that $|E_N| + |E_{N+1}| + \ldots$ is less than δ. Next, choose any finite set of the e_{ji}, with $j > N$, and denote it by $\mathfrak{E}$: $(e_1, e_2, \ldots, e_k)$. Let

$$e = e_1 + e_2 + \ldots + e_k.$$

Then

$$(G^*) \int_e x(\alpha) d\alpha$$

exists and hence

$$(Bk) \int_e x(\alpha) d\alpha$$

exists by the first part of the proof, and by Lemma 6.1 we have

$$\left\| (Bk) \int_e x d\alpha - S(\mathfrak{E}) \right\| < \omega(\mathfrak{E}) < \sum \omega(\mathfrak{E}_j) < \sum \epsilon_j = \tfrac{1}{2}\epsilon.$$

Also, since e is contained in $E_N + E_{N+1} + \ldots$ we have

$$\left\| (G^*) \int_e x(\alpha) d\alpha \right\| = \left\| (Bk) \int_e x(\alpha) d\alpha \right\| < \tfrac{1}{2}\epsilon.$$

Then

$$\begin{aligned} \|S(\mathfrak{E})\| &= \left\| (Bk) \int_e x d\alpha - (Bk) \int_e x d\alpha + S(\mathfrak{E}) \right\| \\ &< \left\| (Bk) \int_e x d\alpha \right\| + \left\| (Bk) \int_e x d\alpha - S(\mathfrak{E}) \right\| \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon. \end{aligned}$$

Hence $\sum_{ij} x(\xi_{ji})|e_{ji}|$ is unconditionally summable. Finally, the diameter of the integral range corresponding to the set of partitions $\mathfrak{E}_j$, which we shall denote by $\mathscr{D}\{\sum_{ij} x(\xi_{ji})\,|e_{ji}|\}$, satisfies the condition

$$\mathscr{D}\{\textstyle\sum_{ij} x(\xi_{ji})\,|e_{ji}|\} < \Sigma\, \omega\,(\mathfrak{E}_j) < \Sigma\, \epsilon_j = \tfrac{1}{2}\epsilon.$$

Since these are the conditions for Birkhoff integrability (1, Theorem 13) the proof of the theorem is complete.

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FUNCTIONS WHICH HAVE GENERALIZED RIEMANN DERIVATIVES

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1. Introduction. Let $f(x)$ be a measurable function defined in the interval (a, b), and let

$$\Delta_n(x, 2h; f) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} f(x + 2jh - nh) \qquad (h > 0; n = 1, 2, \ldots).$$

If the limit of $(2h)^{-n}\,\Delta_n(x, 2h; f)$ exists and is finite at the point x, as $h \to 0$, it is called the nth generalized Riemann derivative of $f(x)$ at the point x, $D^n f(x)$. Considering the upper and lower limits of the above expression we can similarly define the upper and lower nth generalized Riemann derivates, $\bar{D}^n f(x)$ and $\underline{D}^n f(x)$ respectively. If $\bar{D}^n f(x) = \underline{D}^n f(x)$, their common value is the nth generalized Riemann derivative $D^n f(x)$.

If two functions $F(x)$ and $G(x)$ are such that the nth ordinary derivative of $F(x) - G(x)$ is equal to zero then $F(x)$ and $G(x)$ differ by a polynomial of degree at most $n - 1$. The main purpose of this paper is to study the relations between two functions $F(x)$ and $G(x)$ where the nth generalized Riemann derivative of the continuous function $F(x) - G(x)$ is equal to zero, first for derivatives of second order and later for derivatives of higher order.

In the case $n = 2$, if $D^2(F - G) = 0$ then $F(x) - G(x)$ is linear. This follows from Denjoy's work. In order to form a background for a study of the cases in which $n > 2$ we first give a proof for $n = 2$ in conformity with our notations and methods. It turns out that for $n > 2$ additional conditions must be imposed on $F(x) - G(x)$ to ensure that $D^n(F - G) = 0$ makes $F(x) - G(x)$ a polynomial of degree at most $n - 1$. These conditions are considered in §4. Our main result is Theorem 4.2.

2. Definition of the operators H_2 and H_3. Let $F(x)$ be a single valued function defined over a given domain. Then

$$(2.1)\quad H_2(F\colon \alpha, \beta, \gamma) = F(\gamma) - \frac{\gamma - \alpha}{\beta - \alpha} F(\beta) - \frac{\gamma - \beta}{\alpha - \beta} F(\alpha),$$

$$(2.2)\quad H_3(F\colon \alpha, \beta, \gamma, \delta) = F(\delta) - \frac{(\delta - \alpha)(\delta - \beta)}{(\gamma - \alpha)(\gamma - \beta)} F(\gamma) - \frac{(\delta - \alpha)(\delta - \gamma)}{(\beta - \alpha)(\beta - \gamma)} F(\beta) - \frac{(\delta - \beta)(\delta - \gamma)}{(\alpha - \beta)(\alpha - \gamma)} F(\alpha),$$

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where $\alpha, \beta, \gamma, \delta$ are on the domain of $F(x)$, and α, β, γ are distinct points except in the case of H_2 when γ may coincide with α or with β. The operator H_n is defined in §4 (Definition 4.1).

3. The fundamental theorem for $n = 2$. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that $F(x) - G(x)$ is continuous on $[a, b]$, and $D^2(F - G) = 0$ at all points of (a, b), then*

$$H_2(F: x_1, x_2, x_3) = H_2(G: x_1, x_2, x_3)$$

for every three points of $[a, b]$ with $x_1 \neq x_2$.

This theorem has been proved for the case where $F(x)$ and $G(x)$ are both continuous by James (**3**) and by Jeffery (**4**) where use is made of convex functions. In our proof no use is made of convex functions.

In order to prove the fundamental theorem for $n = 2$ we need the following result due to Denjoy (**2**, pp. 18–19). We give a proof in conformity with the notations and methods which we shall use for $n > 2$.

THEOREM 3.1. *Let $\bar{D}^2f(x)$ and $\underline{D}^2f(x)$ be the upper and lower second generalized Riemann derivates of $f(x)$ which is continuous on $[a, b]$. Then, for every three distinct points of $[a, b]$, x_1, x_2, x_3,*

$$\inf_{a<x<b} \bar{D}^2f(x) \leqslant \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)} \leqslant \sup_{a<x<b} \underline{D}^2f(x). \tag{3.1}$$

To establish this theorem we consider the function

$$g(x) = H_2(f: x_1, x_2, x_3, x) \qquad (a < x < b). \tag{3.2}$$

According to (2.2), $g(x_1) = g(x_2) = g(x_3) = 0$. Let us assume $x_1 < x_3 < x_2$. Then, the continuous function $g(x)$ attains a non-negative maximum at some point q of the interval (x_1, x_2); this is obvious if $g(x) > 0$ at one point of (x_1, x_2). The point q may coincide with x_3, as it happens when $g(x) \leqslant 0$ at all points of (x_1, x_2). Consequently

$$[g(q + 2h) - g(q)] - [g(q) - g(q - 2h)] \leqslant 0,$$

whence, according to (2.1)

$$(2h)^{-2} H_2(g: q - 2h, q, q + 2h) \leqslant 0 \tag{3.3}$$

for any h, $0 < 2h < \min\,(q - x_1, x_2 - q)$.

Returning now to (3.2) we can obtain by simple computation

$$\frac{H_2(g: p_1, p_2, p_3)}{(p_3 - p_1)(p_3 - p_2)} = \frac{H_2(f: p_1, p_2, p_3)}{(p_3 - p_1)(p_3 - p_2)} - \frac{H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)} \tag{3.4}$$

where p_1, p_2, p_3 are three arbitrary distinct points. Thus, setting $p_1 = q - 2h$, $p_2 = q$, $p_3 = q + 2h$:

$$(3.5)\quad \frac{H_2(g: q-2h, q, q+2h)}{(2h)^2} = \frac{H_2(f: q-2h, q, q+2h)}{(2h)^2} - \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

Relations (3.3) and (3.5) combine to give

$$(3.6)\quad \frac{H_2(f: q-2h, q, q+2h)}{(2h)^2} < \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

Considering a sequence of h for which the left side of (3.6) tends to the upper second generalized Riemann derivate of $f(x)$ at the point q, $\bar{D}^2 f(q)$, we have

$$\bar{D}^2 f(q) < \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}$$

and consequently

$$(3.7)\quad \inf_{a<x<b} \bar{D}^2 f(x) < \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

By a similar argument dealing with the minimum attained by the continuous function $g(x)$ on the interval (x_1, x_2), we arrive at the relation

$$(3.8)\quad \sup_{a<x<b} \underline{D}^2 f(x) > \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

Relations (3.7) and (3.8) establish Theorem 3.1.

In proving relation (3.1) we assumed $x_1 < x_3 < x_2$. However, (3.1) holds for x_1, x_2, x_3 arbitrary but distinct since the expression

$$\frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}$$

remains invariant under all permutations of x_1, x_2, x_3.

In order to prove the fundamental theorem for $n = 2$ we consider the functions $F(x)$ and $G(x)$, where $F(x) - G(x)$ is continuous on $[a, b]$ and $D^2(F - G) = 0$ at all points of (a, b). Then, by (3.1)

$$\frac{2\,H_2(F - G: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)} = 0$$

for every three distinct points of $[a, b]$, x_1, x_2, x_3. It follows that

$$H_2(F - G: x_1, x_2, x_3) = 0$$

whence, according to (2.1)

$$H_2(F: x_1, x_2, x_3) = H_2(G: x_1, x_2, x_3).$$

4. The fundamental theorem for $n > 2$. The fundamental theorem fails for $n = 3$ as we can easily show by considering two functions $F(x)$ and $G(x)$ that are defined on the interval $[-2, +3]$ and are such that $F(x)$

$-G(x) = |x|$. The function $|x|$ is continuous on $[-2, +3]$ and is such that $D^3|x| = 0$ at all points of $(-2, +3)$. Yet, $H_3(|x|: -1, 0, 1, 2)$ is not zero as can be seen by applying (2.2). Similarly, relation (3.1) fails, because $H_3(|x|: -1, 0, 1, 2) \neq 0$.

As we have mentioned in §1, in order to generalize the fundamental theorem we must impose additional conditions on the difference $F(x) - G(x)$. One procedure is to impose conditions on the generalized Riemann derivatives of $F(x) - G(x)$; in this way, we can show the following:

THEOREM 4.1. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that at every point of (a, b), $D^3(F - G) = 0$, $D^2(F - G)$ exists, and $D^1(F - G)$ exists, then*

$$H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4)$$

for every four points x_1, x_2, x_3, x_4 of $[a, b]$, the first three being distinct.

Remark 4.1. We can make Theorem 4.1 stronger by replacing the existence of $D^2(F - G)$ with the weaker condition that

$$\lim_{h \to 0} (2h)^{-1} H_2(F - G: x - 2h, x, x + 2h) = 0$$

at every point x of (a, b).

LEMMA 4.1. *If $(2h)^{-1} H_2(f: x - 2h, x, x + 2h)$ tends to zero with h for every $x \in (a, b)$, and if $D^1f(x)$ exists at every point x of (a, b), then the derivative exists at every point x of (a, b): in fact*

$$\frac{d}{dx} f(x) = D^1 f(x).$$

It follows that $f(x)$ is continuous on $[a, b]$.

The truth of this lemma follows from the identities

$$+ (4h)^{-1}[f(x + 2h) - 2f(x) + f(x - 2h)]$$
$$= (4h)^{-1}[f(x + 2h) - f(x - 2h)] - (-2h)^{-1}[f(x - 2h) - f(x)],$$

$$- (4h)^{-1}[f(x + 2h) - 2f(x) + f(x - 2h)]$$
$$= (4h)^{-1}[f(x + 2h) - f(x - 2h)] - (2h)^{-1}[f(x + 2h) - f(x)].$$

Taking limits as $h \to 0$ $(h > 0)$, we get

$$0 = D^1 f(x) - \frac{d}{dx} f(x).$$

In order to prove Theorem 4.1 we observe that the function $F(x) - G(x)$ satisfies the conditions of Lemma 4.1, and consequently its derivative exists everywhere on (a, b). Then, according to a theorem of Verblunsky (**6**, p. 393), together with the condition that $D^3(F - G) = 0$ everywhere on (a, b), we conclude that $F(x) - G(x)$ is a polynomial of degree at most 2 on $[a, b]$. We have by direct application of (2.2)

$$H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4).$$

We now determine a set of conditions, different from those of Theorem 4.1, under which $H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4)$.

DEFINITION 4.1. Let $F(x)$ be any single valued function defined over a given domain. Then we define

$$H_n(F: x_1, x_2, \ldots, x_n, x_{n+1}) = w_{n+1}(x_{n+1}) \cdot \sum_{j=1}^{n+1} \frac{F(x_j)}{w'_{n+2}(x_j)} \qquad (n = 0, 1, 2, \ldots),$$

where

$$w_{n+1}(y) = \prod_{i=1}^{n} (y - x_i),$$

and the "primes" denote ordinary differentiations. For $n = 1$, the above relation reads

$$H_1(F: x_1, x_2) = F(x_2) - F(x_1);$$

for $n = 0$, we have $H_0(F: x_1) = F(x_1)$.

Now, let $f(x)$ be defined and continuous on $[a, b]$, and suppose that

$$\inf_{a<x<b} \underline{D}^n f(x) \text{ and } \sup_{a<x<b} \bar{D}^n f(x)$$

are finite.

Set

$$(4.1) \qquad g(x) = H_{n+1}(f: x_1, x_2, \ldots, x_{n+1}, x) \qquad (a < x < b),$$

$$(4.2) \quad y(x, h) = H_{n-2}(g: x - nh + 2h, x - nh + 4h, \ldots, x + nh - 2h) \qquad (h > 0)$$

where $x_1, x_2, \ldots, x_{n+1}$ are $n + 1$ arbitrary points of $[a, b]$ such that $x_1 < x_2 < \ldots < x_n < x_{n+1}$.

It follows directly from (4.1) that $g(x_i) = 0$, where $i = 1, 2, \ldots, n + 1$. Consequently, the continuous function $g(x)$ attains n extrema, each of which is an absolute extremum over one of the intervals (x_j, x_{j+1}), $(j = 1, 2, \ldots, n)$.

Let q be the point of (x_1, x_2) at which $g(x)$ attains its absolute extremum over the interval (x_1, x_2). Then, for h fixed and small, we can find two points x' and x'' of the intervals (x_1, q) and (q, x_2) respectively, such that

$$[g(x' + h) - g(x' - h)]\,[g(x'' + h) - g(x'' - h)] < 0$$

where $x' + h = q = x'' - h$. The function $u_1(x, h) = g(x + h) - g(x - h)$ is continuous in x for h fixed. It follows that $u_1(x, h)$ vanishes at some point of the interval $[x', x'']$ $(x_1 < x' < x'' < x_2)$, because $u_1(x, h)$ changes sign between x' and x''.

Dealing in a similar way with the absolute extrema of $g(x)$ over the remaining $n - 1$ intervals (x_s, x_{s+1}) $(s = 2, 3, \ldots, n)$, we conclude that the function $u_1(x, h)$, for h fixed and small, vanishes at n points of the interval (x_1, x_{n+1}).

Applying successively the same argument to the functions

$$u_t(x, h) = u_{t-1}(x + h, h) - u_{t-1}(x - h, h) \qquad (t = 2, 3, \ldots, n - 2),$$

we conclude that the function $y(x, h) \equiv u_{n-2}(x, h)$, for h fixed and small, vanishes at three distinct points of the interval (x_1, x_{n+1}), L_h, M_h, and N_h with $L_h < M_h < N_h$. It follows then that the function $y(x, h)$, which is continuous in x, attains an absolute non-negative maximum at some point Q_h of the interval (L_h, N_h) if $y(x, h) > 0$ at one point of (L_h, N_h). The point Q_h may coincide with M_h if $y(x, h) \leqslant 0$ at all points of (L_h, N_h). Similarly, $y(x, h)$ attains an absolute non-positive minimum at some point R_h of the interval (L_h, N_h) and $Q_h \neq R_h$. Consequently

$$y(Q_h + 2h, h) - y(Q_h, h) \leqslant 0, \quad y(Q_h - 2h, h) - y(Q_h, h) \leqslant 0$$

whence

$$y(Q_h + 2h, h) - 2y(Q_h, h) + y(Q_h - 2h, h) \leqslant 0 \tag{4.3}$$

for any h, $0 < 2h < \min (Q_h - L_h, N_h - Q_h)$.

Relations (4.2) and (4.3) combine to give

$$(2h)^{-n} H_n(g: Q_h - nh, Q_h - nh + 2h, \ldots, Q_h + nh) \leqslant 0 \qquad (h > 0). \tag{4.4}$$

Dealing in a similar way with the point R_h, we obtain

$$(2h)^{-n} H_n(g: R_h - nh, R_h - nh + 2h, \ldots, R_h + nh) \geqslant 0 \qquad (h > 0) \tag{4.5}$$

for any h, $0 < 2h < \min (R_h - L_h, N_h - R_h)$.

DEFINITION 4.2. The continuous function $f(x)$ belongs to the class K_n of continuous functions if for arbitrary ϵ there exist h' and h'', satisfying (4.4) and (4.5) respectively, such that the expressions

$$\begin{aligned} &(2h')^{-n} H_n(f: Q_{h'} - nh', Q_{h'} - nh' + 2h', \ldots, Q_{h'} + nh') \\ &(2h'')^{-n} H_n(f: R_{h''} - nh'', R_{h''} - nh'' + 2h'', \ldots, R_{h''} + nh'') \end{aligned} \tag{4.6}$$

lie in the interval $[\inf \underline{D}^n f(x) - \epsilon, \sup \bar{D}^n f(x) + \epsilon]$ $(a < x < b)$.

THEOREM 4.2. *If the continuous function $f(x)$ belongs to the class K_n of continuous functions on $[a, b]$, then*

$$\inf_{a<x<b} \underline{D}^n f(x) \leqslant \frac{n!\, H_n(f: x_1, x_2, \ldots, x_{n+1})}{(x_{n+1} - x_1)(x_{n+1} - x_2) \ldots (x_{n+1} - x_n)} \leqslant \sup_{a<x<b} \bar{D}^n f(x) \tag{4.7}$$

for every $n + 1$ distinct points of $[a, b]$, $x_1, x_2, \ldots, x_{n+1}$.

Consider the identity

$$\begin{aligned} \frac{H_n(g: p_1, \ldots, p_{n+1})}{(p_{n+1} - p_1) \ldots (p_{n+1} - p_n)} &= \frac{H_n(f: p_1, \ldots, p_{n+1})}{(p_{n+1} - p_1) \ldots (p_{n+1} - p_n)} \\ &\quad - \frac{H_n(f: x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)} \end{aligned} \tag{4.8}$$

where $p_1, p_2, \ldots, p_{n+1}$ are $n + 1$ arbitrary distinct points. We substitute $p_1 = Q_{h'} - nh'$, $p_2 = Q_{h'} - (n - 2)h', \ldots, p_{n+1} = Q_{h'} + nh'$ and thus we obtain

$$(4.9)\qquad \frac{H_n(g: Q_{h'} - nh', \ldots, Q_{h'} + nh')}{(2h')^n} = \frac{H_n(f: Q_{h'} - nh', \ldots, Q_{h'} + nh')}{(2h')^n} - \frac{n!\, H_n(f: x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)}.$$

Similarly, we substitute in (4.8) $p_1 = R_{h''} - nh''$, $p_2 = R_{h''} - (n-2)h'', \ldots,$ $p_{n+1} = R_{h''} + nh''$, and thus we obtain

$$(4.10)\qquad \frac{H_n(g: R_{h''} - nh'', \ldots, R_{h''} + nh'')}{(2h'')^n} = \frac{H_n(f: R_{h''} - nh'', \ldots, R_{h''} + nh'')}{(2h'')^n} - \frac{n!\, H_n(f: x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)}.$$

Relations (4.4), (4.5), (4.9), (4.10) combine to give

$$\frac{H_n(f: Q_{h'} - nh', \ldots, Q_{h'} + nh')}{(2h')^n} < \frac{n!\, H_n(f: x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)} < \frac{H_n(f: R_{h''} - nh'', \ldots, R_{h''} + nh'')}{(2h'')^n},$$

whence Theorem 4.2 follows because the expressions (4.6) lie in the interval $[\inf \underline{D}^n f(x) - \epsilon, \sup \bar{D}^n f(x) + \epsilon]$ $(a < x < b)$.

THEOREM 4.3. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that $F(x) - G(x)$ belongs to the class K_n, and $D^n(F - G) = 0$ at all points of (a, b), then*

$$H_n(F: x_1, \ldots, x_{n+1}) = H_n(G: x_1, \ldots, x_{n+1})$$

for every $n + 1$ points of $[a, b]$, $x_1, \ldots, x_{n+1}$, where $x_1, \ldots, x_n$ are distinct points.

To prove this theorem we consider the functions $F(x)$ and $G(x)$ where $F(x) - G(x)$ belongs to the class K_n and is such that $D^n(F - G) = 0$ for $a < x < b$. Then according to (4.7) $H_n(F - G: x_1, \ldots, x_{n+1}) = 0$. It then follows from the definition 4.1 that

$$H_n(F: x_1, \ldots, x_{n+1}) = H_n(G: x_1, \ldots, x_{n+1}).$$

5. Additional remarks. Theorem 4.2 reduces to Denjoy's theorem 3.1 for $n = 2$. Indeed, let $f(x)$ be defined and continuous on $[a, b]$ and suppose that

$$\inf_{a<x<b} \underline{D}^2 f(x) \text{ and } \sup_{a<x<b} \bar{D}^2 f(x)$$

are finite. Putting $n = 2$ in relations (4.1) and (4.2) we obtain

$$\begin{aligned} g(x) &= H_2(f: x_1, x_2, x_3, x) \\ & \qquad\qquad\qquad\qquad (a < x < b) \\ y(x, h) &= H_0(g: x) \end{aligned}$$

whence, by the definition 4.1, we have $y(x, h) = g(x)$. Consequently, the roots and the extrema of $y(x, h)$ are independent of h and are identical with those of the function $g(x)$, respectively. The expressions that correspond to (4.6) are obtained by setting $n = 2$, and thus we get

$$(5.1) \qquad \begin{aligned} &(2h)^{-2} H_2(f: Q - 2h, Q, Q + 2h) \\ &(2h)^{-2} H_2(f: R - 2h, R, R + 2h). \end{aligned}$$

Due to the fact that $\bar{D}^2f(Q)$, $\underline{D}^2f(Q)$, $\bar{D}^2f(R)$, $\underline{D}^2f(R)$, lie in the interval $[\inf \underline{D}^2f(x), \sup \bar{D}^2f(x)]$ $(a < x < b)$, it follows that for arbitrary ϵ there exist values of h for which the expressions (5.1) lie in the interval $[\inf \underline{D}^2f(x) - \epsilon, \sup \bar{D}^2f(x) + \epsilon]$ $(a < x < b)$. Consequently, the arbitrary continuous function $f(x)$ belongs to the class K_2 and we conclude that the class K_2 is identical with the class of all continuous functions of x. Thus, Denjoy's theorem 3.1 as well as the fundamental theorem for $n = 2$ are particular cases of the general theorems 4.2 and 4.3, respectively.

Further, it is easy to show that if the function $f(x)$ possesses an nth ordinary derivative or a de La Vallée Poussin derivative of order n, $f_{(n)}(x)$ (**5**, p. 1), at every point x of (a, b), then $f(x)$ belongs to the class K_n of continuous functions. Indeed, in both these cases the function $f(x)$ possesses an nth generalized Riemann derivative $D^nf(x)$ equal to the nth ordinary derivative, or to $f_{(n)}(x)$, respectively, at every point x of (a, b). Moreover, it is known (**1**, p. 207) that in either of these cases the expression $(2h)^{-n} H_n(f: x - nh, x - nh + 2h, \ldots, x + nh)$, where $a < x - nh < x + nh < b$, lies in the interval $[\inf D^nf(x), \sup D^nf(x)]$ $(a < x < b)$. It then follows that the function $f(x)$ belongs to the class K_n.

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ON SOME THEOREMS OF DOETSCH

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1. Introduction. The spaces $\mathfrak{H}_p(\omega)$, $1 < p < \infty$, ω real are defined to consist of those analytic functions $f(s)$, regular for $Re\, s > \omega$ and for which $\mu_p(f;x)$ is bounded for $x > \omega$ where

$$\mu_p(f;x) = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x+iy)|^p\,dy\right\}^{1/p}, \qquad 1 < p < \infty \tag{1.1}$$

and

$$\mu_\infty(f;x) = \sup_{-\infty<y<\infty}|f(x+iy)|. \tag{1.2}$$

These spaces have been extensively studied—for example, see **(2)**, **(4)**.

In particular two results connect these spaces with the theory of Laplace transforms. These are that if $e^{-\omega t}\phi(t) \in L_p(0, \infty)$, $1 < p < 2$, and if f is the Laplace transform of ϕ, that is,

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt, \qquad Re\, s > \omega$$

then $f \in \mathfrak{H}_q(\omega)$ where

$$p^{-1} + q^{-1} = 1, \tag{1.3}$$

and that conversely if $f \in \mathfrak{H}_p(\omega)$, $1 < p < 2$, then $f(s)$ is the Laplace transform of a function ϕ such that $e^{-\omega t}\phi(t) \in L_q(0, \infty)$. For $1 < p < 2$, these two results are due to Doetsch **(2)**, and for $p = 1$ they are trivial. The two results concern the same space if and only if $p = 2$, when they give necessary and sufficient conditions that $f(s)$ be the Laplace transform of a function ϕ such that $e^{-\omega t}\phi(t) \in L_2(0, \infty)$.

Recently the author **(6, 7)** has considered the Laplace transformation of functions of the form $t^\lambda\phi(t)$, $\phi \in L_p(0, \infty)$, $\lambda > -q^{-1}$, and we propose to generalize Doetsch's results so as to deal with functions of this type, though we shall have to restrict λ to be positive. To this end, which is achieved in § 2, we shall first define certain new spaces $\mathfrak{H}_{\lambda,p}(\omega)$, $\lambda > 0$, $1 < p < \infty$, which in a sense are generalizations of the spaces $\mathfrak{H}_p(\omega)$.

In the case $p = 2$ we shall see that we again obtain necessary and sufficient conditions for a representation, and in § 3 we shall relate these results to some previous work of ours and by so doing show that in this case the conditions for the representation can be slightly relaxed.

Doetsch **(2)** has further shown that for $p = 2$ a certain real inversion formula for the Laplace transformation, originally due to Paley and Wiener

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(**5**, p. 39), is very useful. In § 4 we shall show how this formula can be generalized to deal with Laplace transforms of the type mentioned earlier.

2. Generalized spaces. In this section we first define the spaces $\mathfrak{H}_{\lambda,p}$, and then prove two theorems generalizing Doetsch's result.

Definition. $\mathfrak{H}_{0,p}(\omega) \equiv \mathfrak{H}_p(\omega)$. *If* $\lambda > 0$, $\mathfrak{H}_{\lambda,p}(\omega)$ *consists of those functions* $f(s) \in \mathfrak{H}_p(\omega')$ *for every* $\omega' > \omega$ *and such that* $\mu_p^\lambda(f;\omega)$ *is finite, where*

$$\mu_p^\lambda(f;\omega) = \int_\omega^\infty (x-\omega)^{q\lambda-1}(\mu_p(f;x))^q dx, \qquad 1 < p < \infty \tag{2.1}$$

and

$$\mu_1^\lambda(f;\omega) = \sup_{x>\omega} (x-\omega)^\lambda \mu_1(f;x). \tag{2.2}$$

It is clear that $\mathfrak{H}_{\lambda,p}(\omega)$ is a linear space. It is easy to show that it is a Banach space under the norm

$$||f||_{\lambda,p} = \begin{cases} \{\mu_p^\lambda(f;\omega)/\Gamma(\lambda q)\}^{1/q} & \lambda > 0, p > 1 \\ \mu_1^\lambda(f;\omega) & \lambda > 0, p = 1 \\ \sup\limits_{x>\omega} \mu_p(f;x) & \lambda = 0. \end{cases}$$

Also an easy proof shows that if $||f||_{\lambda,p} < M$, $0 < \lambda < \lambda_0$, then $||f||_{0,p} < M$, and $||f||_{\lambda,p} \to ||f||_{0,p}$ as $\lambda \to 0+$. Since these properties are not needed in what ensues, they will not be elaborated further here.

THEOREM 1. *If* $e^{-\omega t}\phi(t) \in L_p(0,\infty)$, $1 < p < 2$, $\lambda > 0$ *and*

$$f(s) = \int_0^\infty e^{-st}\, t^\lambda \phi(t)dt, \qquad Re\, s > \omega,$$

then

$$f(s) \in \mathfrak{H}_{\lambda,q}(\omega).$$

Proof. If $\lambda = 0$, $1 < p < 2$, the theorem follows from (**2**, Theorem 2), and if $\lambda = 0$, $p = 1$, the result is trivial.

If $\lambda > 0$ and $\omega' > \omega$, then since

$$t^\lambda e^{-(\omega'-\omega)t}$$

is bounded for $t > 0$,

$$e^{-\omega' t}t^\lambda\phi(t) \in L_p(0,\infty),$$

and hence by (**2**, Theorem 2) $f(s) \in \mathfrak{H}_q(\omega')$. It remains to show $\mu_q^\lambda(f;\omega)$ is finite.

If $p = 1$, $x > \omega$,

$$|f(x+iy)| < \int_0^\infty e^{-xt}\, t^\lambda |\phi(t)|dt$$

so that

$$\mu_\infty(f;x) < \int_0^\infty e^{-xt}\, t^\lambda |\phi(t)|dt.$$

Hence,

$$\begin{aligned}\mu_\infty^\lambda(f;\omega) &= \int_\omega^\infty (x-\omega)^{\lambda-1}\,\mu_\infty(f;x)dx \\ &< \int_\omega^\infty (x-\omega)^{\lambda-1}dx \int_0^\infty e^{-xt}\,t^\lambda|\phi(t)|dt \\ &= \int_0^\infty t^\lambda|\phi(t)|dt \int_\omega^\infty (x-\omega)^{\lambda-1}\,e^{-xt}\,dx \\ &= \Gamma(\lambda)\int_0^\infty e^{-\omega t}|\phi(t)|dt < \infty,\end{aligned}$$

and $f \in \mathfrak{H}_{\lambda,\infty}(\omega)$.

If $1 < p < 2$, $\lambda > 0$, $x > \omega$,

$$f(x-iy) = \int_0^\infty e^{iyt}(e^{-xt}\,t^\lambda\phi(t))dt$$

is the Fourier transform of a function in $L_p(0,\infty)$, $1 < p < 2$. Hence by (8, Theorem 74), for $x > \omega$,

$$\begin{aligned}\mu_q(f;x) &= \left\{\frac{1}{2\pi}\int_{-\infty}^\infty |f(x+iy)|^q\,dy\right\}^{1/q} = \left\{\frac{1}{2\pi}\int_{-\infty}^\infty |f(x-iy)|^q\,dy\right\}^{1/q} \\ &< \left\{\int_0^\infty e^{-pxt}\,t^{p\lambda}|\phi(t)|^p\,dt\right\}^{1/p},\end{aligned}$$

so that for $x > \omega$,

$$(\mu_q(f;x))^p < \int_0^\infty e^{-pxt}\,t^{p\lambda}|\phi(t)|^p\,dt.$$

Hence, we have

$$\begin{aligned}\mu_q^\lambda(f;\omega) &= \int_\omega^\infty (x-\omega)^{p\lambda-1}\,(\mu_q(f;x))^p\,dx \\ &< \int_\omega^\infty (x-\omega)^{p\lambda-1}\,dx\int_0^\infty e^{-pxt}\,t^{p\lambda}|\phi(t)|^p\,dt \\ &= \int_0^\infty t^{p\lambda}|\phi(t)|^p\,dt\int_\omega^\infty (x-\omega)^{p\lambda-1}\,e^{-pxt}\,dx \\ &= \frac{\Gamma(p\lambda)}{(p)^{p\lambda}}\int_0^\infty e^{-p\omega t}|\phi(t)|^p\,dt < \infty,\end{aligned}$$

and $f \in \mathfrak{H}_{\lambda,q}(\omega)$.

THEOREM 2. *If $f \in \mathfrak{H}_{\lambda,p}(\omega)$, $1 < p < 2$, $\lambda > 0$, then there is a function ϕ with $e^{-\omega t}\phi(t) \in L_q(0,\infty)$ such that*

$$f(s) = \int_0^\infty e^{-st}\,t^\lambda\phi(t)\,dt.$$

Proof. Without loss of generality we may assume $\omega = 0$, for otherwise we deal with $f(\omega + s)$. We shall consider first the cases $1 < p < 2$.

By the definition of $\mathfrak{H}_{\lambda,p}$, $f \in \mathfrak{H}_p(\omega')$ for each $\omega' > 0$, and hence, for each fixed $x > 0$, $f(x + iy) \in L_p(-\infty, \infty)$. For $x > 0$ let

$$F_a(t, x) = \frac{1}{2\pi}\int_{-a}^{a} f(x + iy)e^{ity}\,dy.$$

By (8, Theorem 74), as $a \to \infty$ F_a converges in mean of order q, as a function of t, to a function $F(t, x) \in L_q(-\infty, \infty)$. Consider, however, the integral

$$\int f(s)e^{ts}ds$$

taken around the rectangle with vertices at $x_1 \pm ia$ and $x_2 \pm ia$ where $0 < x_1 < x_2$.

The integral along the upper side is

$$\int_{x_1}^{x_2} f(x + ia)e^{t(x+ia)}dx = e^{ita}\int_{x_1}^{x_2} f(x + ia)e^{tx}\,dx.$$

But if we let $\Phi(\zeta) = f(\omega' - i\zeta)$, where $0 < \omega' < x_1$, we have that $\Phi(\zeta)$ is an analytic function regular for $\eta = Im\,\zeta > 0$, and for $\eta > 0$,

$$\int_{-\infty}^{\infty} |\Phi(\xi + i\eta)|^p\,d\xi = \int_{-\infty}^{\infty} |f(\omega' + \eta - i\xi)|^p\,d\xi$$
$$= \int_{-\infty}^{\infty} |f(\omega' + \eta + i\xi)|^p\,d\xi = 2\pi(\mu_p(f; \omega' + \eta))^p,$$

and this is bounded for $\eta > 0$ since $f \in \mathfrak{H}_p(\omega')$. Hence, by (8, Lemma, p. 125), $\Phi(\xi + i\eta) \to 0$ as $\xi \to -\infty$ uniformly for $\delta < \eta < R$ where $R > \delta > 0$. Taking $\xi = -a$, $\eta = x - \omega'$, $R = x_2 - \omega'$, $\delta = x_1 - \omega'$, we have $f(x+ia) \to 0$ as $a \to \infty$ uniformly for $x_1 < x < x_2$, and the integral along the upper side of the rectangle tends to zero as $a \to \infty$. Similarly the integral along the lower side tends to zero as $a \to \infty$. Hence, as $a \to \infty$,

$$\int_{-a}^{a} f(x_1 + iy)e^{t(x_1+iy)}dy - \int_{-a}^{a} f(x_2 + iy)e^{t(x_2+iy)}dy \to 0,$$

that is,

$$e^{tx_1}F_a(t, x_1) - e^{tx_2}F_a(t, x_2) \to 0.$$

Thus the mean limit over any finite t-interval is also zero, so that for almost all t

$$e^{tx_1}F(t, x_1) = e^{tx_2}F(t, x_2),$$

and we may write

$$F(t, x) = e^{-tx}F(t).$$

By (8, Theorem 74)

$$\int_{-\infty}^{\infty} |F(t)|^q\,e^{-qxt}\,dt < (\mu_p(f; x))^q. \tag{2.3}$$

Since for any $\delta > 0$ the right hand side of (2.3) is bounded, say by $K(\delta)$, for $x > \delta$, we have

$$\int_{-\infty}^{-\delta} |F(t)|^q \, dt < e^{-q\delta x} \int_{-\infty}^{\infty} |F(t)|^q \, e^{-qxt} \, dt$$
$$< K(\delta) \, e^{-q\delta x} \to 0 \quad \text{as} \quad x \to \infty.$$

Thus $F(t) = 0$ a.e. for $t < 0$, and (2.3) becomes

$$\int_0^{\infty} |F(t)|^q \, e^{-qxt} \, dt < (\mu_p(f; x))^q. \tag{2.4}$$

Multiplying (2.4) by $x^{q\lambda-1}$ and integrating, we obtain

$$\frac{\Gamma(q\lambda)}{q^{q\lambda}} \int_0^{\infty} t^{-q\lambda} |F(t)|^q \, dt < \int_0^{\infty} x^{q\lambda-1} (\mu_p(f; x))^q \, dx$$
$$= \mu_p^{\lambda}(f; 0) < \infty,$$

so that $t^{-\lambda} F(t) \in L_q(0, \infty)$, or $F(t) = t^{\lambda} \phi(t)$, where $\phi \in L_q(0, \infty)$. Finally, from **(8, Theorem 74)**, for $x > 0$ and almost all y,

$$f(x + iy) = \frac{d}{dy} \int_0^{\infty} \frac{e^{-iyt} - 1}{-it} \, e^{-xt} \, t^{\lambda} \phi(t) \, dt$$
$$= \frac{d}{dy} \int_0^{\infty} e^{-xt} \, t^{\lambda} \phi(t) dt \int_0^{y} e^{-itu} \, du$$
$$= \frac{d}{dy} \int_0^{y} du \int_0^{\infty} e^{-(x+iu)t} \, t^{\lambda} \phi(t) \, dt$$
$$= \int_0^{\infty} e^{-(x+iy)t} \, t^{\lambda} \phi(t) \, dt,$$

the interchange of the order of integrations being justified by Fubini's theorem. But since the functions appearing on either side of this equation are continuous, the equation holds for all y and thus, if $Re\, s > 0$,

$$f(s) = \int_0^{\infty} e^{-st} \, t^{\lambda} \phi(t) \, dt.$$

For $p = 1$ we proceed as follows. By the definition of $\mathfrak{H}_{\lambda,1}$, $f \in \mathfrak{H}_1(\omega')$ for any $\omega' > 0$, and thus for each $x > 0$, $f(x + iy) \in L_1(-\infty, \infty)$. For $x > 0$ we let

$$F(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x + iy) e^{ity} \, dy.$$

Then it follows in practically the same manner as previously that for almost all t

$$F(t, x) = e^{-tx} \, F(t).$$

Hence,

$$e^{-tx} |F(t)| < \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x + iy)| dy = \mu_1(f; x), \tag{2.5}$$

and since the right hand side of (2.5) is bounded as $x \to \infty$, we must have $F(t) = 0$ for $t < 0$.

Multiplying both sides of (2.5) by x^λ and taking suprema, we obtain

$$\sup_{x>0} x^\lambda e^{-tx}|F(t)| < \mu_1^\lambda(f;0).$$

But

$$\sup_{x>0} x^\lambda e^{-tx} = \lambda^\lambda e^{-\lambda} t^{-\lambda},$$

so that

$$t^{-\lambda}|F(t)| < M, \qquad t > 0,$$

that is $F(t) = t^\lambda \phi(t)$ with $\phi \in L_\infty(0, \infty)$.

Finally from (8, Theorem 3), for $x > 0$

$$f(x + iy) = \lim_{R\to\infty} \int_0^R e^{-iyt} e^{-xt} t^\lambda \phi(t)\, dt = \int_0^\infty e^{-(x+iy)t} t^\lambda \phi(t)\, dt,$$

so that for $Re\, s > 0$

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t)\, dt.$$

3. The case $p = 2$. Theorems 1 and 2 together give for $p = 2$ necessary and sufficient conditions that $f(s)$ be represented as the Laplace transform of a function of the form $t^\lambda \phi(t)$ with $e^{-\omega t}\phi(t) \in L_2(0, \infty)$ and $\lambda > 0$. However, these conditions can be somewhat relaxed by using a previous result of ours. This is done in the following theorem. For convenience we write here $\lambda = \frac{1}{2}\nu$.

THEOREM 3. *A necessary and sufficient condition that an analytic function $f(s)$, regular for $Re\, s > \omega$ be the Laplace transform of a function of the form $t^{\frac{1}{2}\nu}\phi(t)$, with $e^{-\omega t}\phi(t) \in L_2(0, \infty)$, ω real, $\nu > 0$, is that*

$$\int_\omega^\infty (x - \omega)^{\nu-1} dx \int_{-\infty}^\infty |f(x + iy)|^2 dy < \infty.$$

Proof. We may suppose, without loss of generality, that $\omega = 0$. In (7) we showed that a necessary and sufficient condition for such a representation is that

$$\sum_{n=0}^\infty \frac{n!}{\Gamma(\nu + n + 1)} |q_n|^2 < \infty, \tag{3.1}$$

where

$$q_n = \sum_{r=0}^n \binom{n+\nu}{n-r} \frac{1}{r!} f^{(r)}(\tfrac{1}{2}).$$

We shall show here that the two conditions are equivalent.

Now, if $\nu > 0$,

$$\frac{n!}{\Gamma(\nu + n + 1)} = \frac{B(n + 1, \nu)}{\Gamma(\nu)} = \frac{2}{\Gamma(\nu)} \int_0^1 r^{2n+1}(1 - r^2)^{\nu-1} dr,$$

and hence (3.1) becomes

$$\frac{2}{\Gamma(\nu)}\int_0^1 r(1-r^2)^{\nu-1}\left(\sum_{n=0}^{\infty}|q_n|^2 r^{2n}\right)dr < \infty,$$

the interchange of integration and summation being permitted since all summands are positive.

But it was pointed out in (7) that

$$\sum_{n=0}^{\infty} q_n z^n = \frac{F(z)}{(1-z)^{\nu+1}}, \quad |z| < 1,$$

where $F(z) = f(\frac{1}{2}(1+z)/(1-z))$. Hence, from the Parseval theorem for power series (3, p. 245), for $0 < r < 1$

$$\sum_{n=0}^{\infty}|q_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{F(re^{i\theta})}{(1-re^{i\theta})^{\nu+1}}\right|^2 d\theta$$

and (3.1) becomes

$$\frac{1}{\pi\Gamma(\nu)}\int_0^1 (1-r^2)^{\nu-1} r\,dr\int_0^{2\pi}\left|\frac{F(re^{i\theta})}{(1-re^{i\theta})^{\nu+1}}\right|^2 d\theta < \infty. \tag{3.2}$$

However, the transformation

$$re^{i\theta} = z = \frac{s-\frac{1}{2}}{s+\frac{1}{2}} = \frac{x+iy-\frac{1}{2}}{x+iy+\frac{1}{2}}$$

maps the interior of the unit circle in the z-plane conformally and univalently onto the half-plane $Re\, s > 0$, and making this change of variable in the integral, (3.2) becomes

$$\frac{2^{\nu-1}}{\pi\Gamma(\nu)}\int_0^{\infty} x^{\nu-1}dx\int_{-\infty}^{\infty}|f(x+iy)|^2 dy < \infty,$$

that is, the condition of the theorem.

It is worth noting the points in which the conditions are relaxed here. Using Theorems 1 and 2 we obtain the condition $f \in \mathfrak{H}_{\lambda,2}(\omega)$ as necessary and sufficient for such a representation. From the definition of $\mathfrak{H}_{\lambda,2}(\omega)$, this implies $f \in \mathfrak{H}_2(\omega')$ for every $\omega' > \omega$, that is, that

$$\int_{-\infty}^{\infty}|f(x+iy)|^2 dy$$

be bounded for $x > \omega'$, for every $\omega' > \omega$, and it is this condition that is dropped. It may also be noted that Theorems 1 and 3 together imply that the condition $f \in \mathfrak{H}_2(\omega')$ for each $\omega' > \omega$, can be dropped from the definition of $\mathfrak{H}_{\lambda,2}(\omega')$.

It is natural to ask whether the condition $f \in \mathfrak{H}_p(\omega')$ for each $\omega' > \omega$ can be dropped from the definition of $\mathfrak{H}_{\lambda,p}(\omega)$ for other values of p. For $p = 1$ and $p = \infty$ this question can be answered affirmatively. In the case $p = 1$, this follows from the fact that for $x > \omega' > \omega$,

$$\mu_1(f;x) < (\omega'-\omega)^{-\lambda}\,\mu_1^{\lambda}(f;\omega),$$

and for $p = \infty$ the affirmative answer can easily be shown to follow from a theorem of Doetsch **(1)** which asserts that $\log \mu_\infty(f;x)$ is a convex function of x. For the remaining values of p the answer is not yet known.

4. Inversion for $p = 2$. The inversion theorem is proved below. We first prove a preliminary lemma.

LEMMA. *Suppose $\phi \in L_2(0, \infty)$, $\lambda > 0$, and*

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t)\, dt, \qquad s > 0.$$

Then for $s > 0$

$$\frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma) d\sigma = \int_0^\infty e^{-st}\phi(t) dt.$$

Proof.

$$\begin{aligned}\frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma) d\sigma &= \frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} d\sigma \int_0^\infty e^{-\sigma t} t^\lambda \phi(t)\, dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^\lambda \phi(t)\, dt \int_s^\infty (\sigma - s)^{\lambda-1} e^{-\sigma t}\, d\sigma \\ &= \int_0^\infty e^{-st}\phi(t)\, dt,\end{aligned}$$

the interchange of the orders of integration being justified by Fubini's theorem.

THEOREM 4. *If $\phi \in L_2(0, \infty)$, $\lambda > 0$, and*

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t) dt, \qquad Re\, s > 0,$$

then

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}}\, \frac{t^{-\lambda}}{\pi} \int_0^\infty f(s) E_\lambda(st, \alpha)\, ds,$$

where for $x > 0$,

$$E_\lambda(x, \alpha) = \int_0^\alpha Re\left\{\frac{x^{\lambda-\frac{1}{2}+iy}}{\Gamma(\lambda + \frac{1}{2} + iy)}\right\} dy.$$

Proof. For $\lambda = 0$ the result is given in **(2**, Theorem 6**)**. We shall deduce the result for $\lambda > 0$ from that for $\lambda = 0$. For this suppose $\lambda > 0$. Then by the lemma

$$\frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma)\, d\sigma$$

is the Laplace transform of ϕ, and hence

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}}\, \frac{1}{\pi\Gamma(\lambda)} \int_0^\infty E_0(st, \alpha)\, ds \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma)\, d\sigma. \tag{4.1}$$

But since the theorem is true for $\lambda = 0$, it follows that if $g(s)$ is the Laplace transform of a function in $L_2(0, \infty)$, then for all sufficiently large α

$$\int_0^\infty |E_0(st, \alpha)g(s)|ds < \infty.$$

Also, as in the proof of the lemma, if $s > 0$

$$\int_s^\infty (\sigma - s)^{\lambda-1}|f(\sigma)|d\sigma < \int_s^\infty (\sigma - s)^{\lambda-1}d\sigma \int_0^\infty e^{-\sigma t}t^\lambda |\phi(t)|dt$$
$$= \int_0^\infty e^{-st}|\phi(t)|dt = \bar{g}(s),$$

and thus since $|\phi(t)| \in L_2(0, \infty)$, we have for all sufficiently large α

$$\int_0^\infty |E_0(st, \alpha)|ds \int_s^\infty (\sigma - s)^{\lambda-1}|f(\sigma)|d\sigma < \int_0^\infty |E_0(st, \alpha)\bar{g}(s)|ds < \infty.$$

Hence by Fubini's theorem we may interchange the order of integrations in equation (4.1) and obtain

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{1}{\pi\Gamma(\lambda)} \int_0^\infty f(\sigma)d\sigma \int_0^\sigma (\sigma - s)^{\lambda-1}E_0(st, \alpha)ds. \tag{4.2}$$

However,

$$\frac{1}{\Gamma(\lambda)} \int_0^\sigma (\sigma - s)^{\lambda-1}E_0(st, \alpha)ds$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\sigma (\sigma - s)^{\lambda-1}ds \int_0^\alpha Re\left\{\frac{(st)^{-\frac{1}{2}+iy}}{\Gamma(\frac{1}{2} + iy)}\right\} dy$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\alpha Re\left\{\frac{t^{-\frac{1}{2}+iy}}{\Gamma(\frac{1}{2} + iy)} \int_0^\sigma (\sigma - s)^{\lambda-1}s^{-\frac{1}{2}+iy}ds\right\} dy$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\alpha Re\left\{\frac{t^{-\frac{1}{2}+iy}}{\Gamma(\frac{1}{2} + iy)} \sigma^{\lambda-\frac{1}{2}+iy}B(\lambda, \tfrac{1}{2} + iy)\right\} dy$$
$$= t^{-\lambda} \int_0^\alpha Re\left\{\frac{(\sigma t)^{\lambda-\frac{1}{2}+iy}}{\Gamma(\lambda + \frac{1}{2} + iy)}\right\} dy = t^{-\lambda}E_\lambda(st, \alpha).$$

Hence (4.2) becomes

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{t^{-\lambda}}{\pi} \int_0^\infty f(s)E_\lambda(st, \alpha)ds.$$

COROLLARY. *If* $e^{-\omega t}\phi(t) \in L_2(0, \infty)$, $\lambda > 0$, *and*

$$f(s) = \int_0^\infty e^{-st}t^\lambda \phi(t)dt, \qquad Re\, s > \omega,$$

then

$$\phi(t) = e^{\omega t} \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{t^{-\lambda}}{\pi} \int_\omega^\infty f(s)E_\lambda((s - \omega)t, \alpha)ds.$$

Proof. The result follows on applying the theorem to $f(s + \omega)$, which is the Laplace transform of $t^\lambda e^{-\omega t}\phi(t)$.

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SPECTRAL THEORY FOR THE DIFFERENTIAL EQUATION $Lu = \lambda Mu$

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Introduction. Let L and M be linear ordinary differential operators defined on an interval I, not necessarily bounded, of the real line. We wish to consider the expansion of arbitrary functions in eigenfunctions of the differential equation $Lu = \lambda Mu$ on I. The case where M is the identity operator and L has a self-adjoint realization as an operator in the Hilbert space $L^2(I)$ has been treated in various ways by several authors; an extensive bibliography may be found in **(4)** or **(8)**. A characterization of the self-adjoint realizations of L by means of boundary conditions has been given by Kodaira **(11)** and Coddington **(3)**. The elementary approach used by Coddington and Levinson **(4**; chap. 10**)** has been used by the author in **(1)** to show the existence of eigenfunction expansions in the general case, provided M is a positive, semi-bounded operator. Here, a different existence proof is given, based on the spectral theorem and the theory of direct integrals of von Neumann **(12)**. The method, first used by Gårding **(6)** in the special case mentioned above, can be applied to the case where L and M are elliptic partial differential operators, but the results obtained are not as general as those of Gelfand and Kostyucenko **(9)**, Browder **(2)**, and Gårding **(8)**.

Following the development of Gårding **(7, 8)**, the existence of a Green's function is shown, and the analogue of the formula due to Titchmarsh **(14)** and Kodaira **(11)**, relating the Green's function to the spectral matrix, is obtained. Finally, the self-adjoint realizations are studied, and are characterized by boundary conditions.

The author is indebted to Professor Lars Gårding for suggesting the topic of this investigation, as well as for many valuable suggestions in the course of his work.

1. The spectral theorem in direct integral form. Let σ be a measure on the real line R, and let $\nu(\lambda)$ be a function defined for all real λ, taking the values $1, 2, \ldots,$ and ∞, and measurable with respect to σ. Consider vector-valued functions $F(\lambda) = [F_1(\lambda), F_2(\lambda), \ldots]$ with $\nu(\lambda)$ complex-valued components. Let $L^2(\sigma, \nu)$ be the set of all such functions with measurable components for which the square norm

$$(F, F) = \int_R |F(\lambda)|^2 d\sigma(\lambda), \quad \text{where} \quad |F(\lambda)|^2 = \sum_{k=1}^{\nu(\lambda)} |F_k(\lambda)|^2,$$

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is finite. The equivalence classes of $L^2(\sigma, \nu)$ with respect to functions of vanishing norm form a separable Hilbert space. We identify functions in the same equivalence class, and thus refer to $L^2(\sigma, \nu)$ itself as a Hilbert space, with inner product $(F, G) = \int_R F(\lambda)\bar{G}(\lambda)d\sigma(\lambda)$, where

$$F(\lambda)\bar{G}(\lambda) = \sum_{k=1}^{\nu(\lambda)} F_k(\lambda)\bar{G}_k(\lambda).$$

Since the value $F(\lambda)$ of F at the point λ is an element of a Hilbert (sequence) space H_λ of dimension $\nu(\lambda)$, we may regard $L^2(\sigma, \nu)$ as a symbolic integral of the H_λ with respect to σ, called a direct integral. The formula

$$F(\lambda)\bar{G}(\lambda) = \sum_{k=1}^{\nu(\lambda)} F_k(\lambda)\bar{G}_k(\lambda)$$

means that each H has been referred to an orthonormal basis. This has been done for convenience only, and other bases will be used in section 3.

Let A be a self-adjoint linear operator on a separable Hilbert space H. Then a form of the spectral theorem due to von Neumann **(12)** states that there is a direct integral $L^2(\sigma, \nu)$ and a unitary mapping U from H to $L^2(\sigma, \nu)$ which diagonalizes A in the sense that UAU^{-1} is multiplication by λ in $L^2(\sigma, \nu)$. More precisely, if D_A is the domain of A, then $U(D_A)$ consists of those $F \in L^2(\sigma,\nu)$ for which $\lambda F(\lambda) \in L^2(\sigma, \nu)$, that is, for which $\int_R \lambda^2 |F(\lambda)|^2 d\sigma(\lambda) < \infty$, and if $F \in U(D_A)$, then $UAU^{-1}F(\lambda) = \lambda F(\lambda)$ for almost all λ. The measure σ is concentrated on the spectrum of A, and the number $\nu(\lambda)$ may be called the multiplicity of the point λ in the spectrum of A. If λ is an eigenvalue of A, then σ has a jump at λ, and there are $\nu(\lambda)$ linearly independent eigenfunctions of A at λ. If $L^2(\sigma_1, \nu_1)$ and $L^2(\sigma_2, \nu_2)$ are two direct integrals which diagonalize A in this manner, then σ_1 and σ_2 are equivalent (have the same null sets), and $\nu_1 = \nu_2$ except on a set of σ-measure zero.

2. The eigenfunction expansion theorem. Let I be an interval on the real line, not necessarily bounded. Let $L^2(I)$ be the set of all complex-valued square integrable functions on I, with inner product $(f, g) = \int_I f(x)\bar{g}(x)dx$. Let L and $\tilde{M}$ be linear ordinary differential operators of orders n and m respectively $(n > m)$, defined by

$$Lu = \sum_{i=0}^{n} p_i(x)u^{(n-i)}, \quad \tilde{M}u = \sum_{i=0}^{m} q_i(x)u^{(m-i)}.$$

We assume that the p_i and q_i are complex-valued functions of class C^{n-i} and C^{m-i} respectively on I, and that neither p_0 nor q_0 vanishes on any compact subinterval of I.

Let $C^\infty_0(I)$ denote the set of complex-valued functions of class C^∞ on I which vanish identically outside a compact subinterval of I. We will always assume that L and $\tilde{M}$ are symmetric, $(Lf, g) = (f, Lg)$ and $(\tilde{M}f, g) = (f, \tilde{M}g)$ for all $f, g \in C^\infty_0(I)$. This implies that L and $\tilde{M}$ coincide with their formal

adjoints. We will also assume that $\tilde{M}$ is semi-bounded with a positive lower bound, $(\tilde{M}f,f) > c(f,f)$ for some constant $c > 0$ and all $f \in C^n{}_0(I)$. The symmetry and semi-boundedness of $\tilde{M}$ imply that $[f,g] = (\tilde{M}f,g) = (f,\tilde{M}g)$ may be considered as an inner product on $C^n{}_0(I)$. Let H be the Hilbert space completion of $C^n{}_0(I)$ in this inner product. It is clear that H contains all functions whose derivatives up to order m belong to $L^2(I)$ and which vanish identically outside a compact subinterval of I. It is easy to verify that H can be identified with a linear subset of $L^2(I)$. Considered as an operator in the Hilbert space $L^2(I)$, $\tilde{M}$ has at least one self-adjoint extension. There is a unique self-adjoint extension M of $\tilde{M}$ whose domain D_M is contained in H. The range of M is $L^2(I)$, and M has a bounded inverse mapping $L^2(I)$ into H. This result is due to Friedrichs **(5)**; see also **(13**; pp. 331–334**)**. We consider $M^{-1}L$ as an operator in H with domain $C^n{}_0(I)$. This operator is symmetric since $[M^{-1}Lf,g] = (Lf,g) = (f,Lg) = [f,M^{-1}Lg]$ for $f,g \in C^n{}_0(I)$. We assume that $M^{-1}L$ has a self-adjoint extension A, considered as an operator in H. If M and L have real coefficients, then $M^{-1}L$ is a real operator and always has at least one self-adjoint extension **(13**; p. 329**)**. Also, if L is semi-bounded in $L^2(I)$, $(Lf,f) > d(f,f)$ for some constant d and all $f \in C^n{}_0(I)$, then $M^{-1}L$ is a semi-bounded operator in H. Decreasing the lower bound if necessary, we may assume $d < 0$. Since $[f,f] > c(f,f)$, $d/c[f,f] < d(f,f)$, and $[M^{-1}Lf,f] = (Lf,f) > d(f,f) > d/c[f,f]$ for $f \in C^n{}_0(I)$, and $M^{-1}L$ is semi-bounded. Then by the theorem of Friedrichs cited above, $M^{-1}L$ has at least one self-adjoint extension.

We can now state the main result to be obtained in this section:

THEOREM 1: *Let A be a self-adjoint extension of $M^{-1}L$, considered as an operator in H. The spectral theorem furnishes a direct integral $L^2(\sigma,\nu)$ and a unitary transformation U from H to $L^2(\sigma,\nu)$ which diagonalizes A. Under the conditions imposed above on L and M, this transformation is given by $(Uf)(\lambda) = \int_I Mf(x)\bar{E}(x,\lambda)dx$ for $f \in D_M$ and its inverse by $(U^{-1}F)(x) = \int_R F(\lambda)E(x,\lambda)d\sigma(\lambda)$ for $F \in L^2(\sigma,\nu)$, with the integrals converging to the functions in the norms of the Hilbert spaces $L^2(\sigma,\nu)$ and H respectively. The components of $E(x,\lambda)$ are linearly independent functions for almost all λ, having locally square integrable derivatives with respect to x, and are improper eigenfunctions (not necessarily belonging to H) of the differential equation $Lu = \lambda Mu$ for almost all λ. If λ_0 is an eigenvalue of $Lu = \lambda Mu$, then the components of $E(x,\lambda_0)$ are proper eigenfunctions.*

It is well known **(4**, pp. 190–191**)** that there exists a fundamental solution $k(x,y)$ for the operator L with the following properties:

(i) The function k and its partial derivatives up to total order $(n-2)$ are continuous on the square IxI. The partial derivatives of orders $(n-1)$ and n are continuous except on the diagonal $x = y$, and the partial derivatives of order $(n-1)$ have a jump of $1/p_0(y)$ on $x = y$.

(ii) As a function of x, k satisfies $Lu = 0$ if $x \neq y$.

(iii) If S is any compact subinterval of I and f is any function belonging to $C^n{}_0(I)$, then $f(x) = \int_S k(x, y)Lf(y)dy$ for $x \in S$.

Let S be any compact subinterval of I and let $\phi_S \in C^n{}_0(I)$ be equal to 1 on S. If $f \in C^n{}_0(S)$, then $f(x) = \int_S k(x, y)Lf(y)dy = \int_S k(x, y)\phi_S(y)Lf(y)dy$ for $x \in S$. The function $k(x, \ )\phi_S(\ )$, for any fixed $x \in S$, vanishes identically outside some compact subinterval of I and has square integrable derivatives up to order $n - 1 > m$. Thus $k(x, \ )\phi_S(\ )$ belongs to H and $f(x) = (Lf, \bar{k}(x, \ )\bar{\phi}_S(\ )) = [Af, \bar{k}(x, \ )\bar{\phi}_S(\ )]$.

Let $L^2(\sigma, \nu)$ be a suitable direct integral and let U be the unitary mapping of H to $L^2(\sigma, \nu)$ which diagonalizes the self-adjoint operator A. The fact that U is unitary is expressed by the Parseval formula,

$$[f, g] = (Uf, Ug) = \int_R (Uf)(\lambda)(U\bar{g})(\lambda)d\sigma(\lambda)$$

for any $f, g \in H$. Here, and in all that follows, an expression of the form $U\bar{g}$, where U is an operator and g is a function, will denote the complex conjugate of Ug. Let $f \in C^n{}_0(S)$, and let g belong to D^S, the set of functions g in D_M such that Mg vanishes identically outside S. We let $F = Uf$, $G = Ug$, $K(x, \ ) = U\{k(x, \ )\phi_S(\ )\}$, $E(x, \lambda) = \lambda K(x, \lambda)$. Then

$$f(x) = [Af, \bar{k}(x, \ )\bar{\phi}_S(\ )] = (UAf, \bar{K}(x, \ )) = (\lambda Uf, \bar{K}(x, \ )) = (F, \bar{E}(x, \ )),$$

or $f(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)$. In addition

$$\begin{aligned}[f, g] = (f, Mg) &= \int_S[\int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)]M\bar{g}(x)dx \\ &= \int_R F(\lambda)[\int_S M\bar{g}(x)E(x, \lambda)dx]d\sigma(\lambda),\end{aligned}$$

the interchange in the order of integration being justified by the absolute convergence of the integral. On the other hand, $[f, g] = \int_R F(\lambda)\bar{G}(\lambda)d\sigma(\lambda)$, and thus $G(\lambda) = \int_S Mg(x)\bar{E}(x, \lambda)dx$ for almost all λ.

The Parseval formula gives $\|k(x, \ )\phi_S(\ )\|^2 = \int_R|K(x, \lambda)|^2d\sigma(\lambda)$, so that $\int_S\int_R|K(x, \lambda)|^2d\sigma(\lambda)dx < \infty$. By the Fubini theorem, $\int_R\int_S|K(x, \lambda)|^2dxd\sigma(\lambda) < \infty$, and $\int_S|K(x, \lambda)|^2dx$ is finite except for λ in a set of σ-measure zero. Then $\int_S|E(x, \lambda)|^2dx$ is finite if λ is outside this same null set. We can redefine E without changing its equivalence class in $L^2(\sigma, \nu)$ by making $E(x, \lambda) = 0$ when λ is in this null set, and then $\int_S|E(x, \lambda)|^2dx$ is finite for all λ.

For $g \in C^n{}_0(S)$, we have seen that $(Ug)(\lambda) = \int_S Mg(x)\bar{E}(x, \lambda)dx$. Since $MAg = Lg$, which vanishes identically outside S, $Ag \in D^S$, and the above relation holds with g replaced by Ag, so that $(UAg)(\lambda) = \int_S Lg(x)\bar{E}(x, \lambda)dx$. Since $(UAg)(\lambda) = \lambda(Ug)(\lambda)$ for almost all λ, $\int_S[Lg(x) - \lambda Mg(x)]\bar{E}(x, \lambda)dx = 0$ when λ does not belong to a set N_g of σ-measure zero, with N_g dependent on g. The same is true for a sequence g_j of functions when λ does not belong to the null set

$$N = \bigcup_{j=1}^{\infty} N_{g_j}.$$

We choose the sequence g_j dense in $C^\infty_0(S)$, and then $\int_S[Lg(x) - \lambda Mg(x)] \bar{E}(x, \lambda)dx = 0$ for all $g \in C^\infty_0(S)$ if $\lambda \notin N$. We let $E(x, \lambda) = 0$ if $\lambda \in N$, and then this relation holds for all λ. Thus the components of $E(x, \lambda)$ are weak solutions of $Lu = \lambda Mu$ on S. It follows from a well-known theorem on weak solutions of partial differential equations that the components E_k of $E(x, \lambda)$ have derivatives which are locally square integrable, that each E_k is of class C^∞ in x after correction on a null set for each λ, and that $LE_k(x, \lambda) = \lambda ME_k(x,\lambda)$ for $k = 1, 2, \ldots, \nu(\lambda)$ and almost all λ. This theorem is easily proved by using the properties of the fundamental solution; see for example **(10)**.

The components of E depend on the compact subinterval S. Let E' be another function with the same properties, corresponding to another subinterval $S' \supset S$. Then $\int_S Mg(x)[\bar{E}_k(x, \lambda) - \bar{E}'_k(x, \lambda)]dx = 0$ for each k and almost all λ, independent of $g \in D^S$. Since $M(D^S)$ is easily seen to be dense in $L^2(S)$, we may let Mg run through a countable dense subset of $L^2(S)$, and it follows that the components of $E(x, \lambda)$ and $E'(x, \lambda)$ are square integrable in x over S and are equal for λ outside some null set P. We set $E(x, \lambda) = E'(x, \lambda) = 0$ for $\lambda \in P$, and then $E(x, \lambda) = E'(x, \lambda)$ for $x \in S$ and all λ. By taking a sequence of compact subintervals S tending to I, we can extend E uniquely to a function defined for all $x \in I$ and all λ. Each component of E is differentiable with respect to x, and its derivatives are locally square integrable. Also, $LE_k(x, \lambda) = \lambda ME_k(x, \lambda)$ for $k = 1, \ldots, \nu(\lambda)$, and the E_k are improper eigenfunctions of $Lu = \lambda Mu$, not necessarily belonging to the space H.

If λ_0 is an eigenvalue of A, then σ has a jump, which we may assume to be a jump of 1, at λ_0. We choose $F = 0$ except at λ_0, and $F_j(\lambda_0) = \delta_{jk}$ for any fixed index $k \leq \nu(\lambda_0)$. Then $F \in H$ and $(U^{-1}F)(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda) = E_k(x, \lambda_0)$ belongs to H. Thus, if λ_0 is an eigenvalue of A, the $E_k(x, \lambda_0)$ are proper eigenfunctions of $Lu = \lambda Mu$.

The eigenfunctions $E_k(x, \lambda)[k = 1, \ldots, \nu(\lambda)]$ are linearly independent for almost all λ. To prove this, consider the matrices $Q(\lambda) = (q_{jk}(\lambda)) = (\int_S E_j(x, \lambda) \bar{E}_k(x, \lambda)dx)$, where S is a compact subinterval of I. Let $\mu = \mu(\lambda)$ be the rank of $Q(\lambda)$, with the value $\mu = \infty$ being permitted. Then $\mu(\lambda) \leq \nu(\lambda)$, and the $E_k(x, \lambda)$ are linearly independent if and only if $\mu(\lambda) = \nu(\lambda)$. We define the function $F \in L^2(\sigma, \nu)$ as follows. If $\mu(\lambda) = \nu(\lambda)$, we set $F(\lambda) = 0$; otherwise, we set $F_r(\lambda) = 0$ if $r > \mu$ and make $F_r(\lambda)$ proportional to the cofactor of $q_{\mu r}(\lambda)$ in the $\mu \times \mu$ determinant of the $q_{jk}(\lambda)$ for $j, k \leq \mu$ if $r \leq \mu$. The proportionality factor can always be chosen to be non-zero and such that $F \in L^2(\sigma, \nu)$. Now $F(\lambda)E(x, \lambda) = 0$ for all λ. It follows that $\int_R F(\lambda)\bar{G}(\lambda)d\sigma(\lambda) = 0$ if $G = Ug$ and g is in some D^S. Thus $F = 0$ except for λ in a fixed null set, and the $E_k(x, \lambda)$ are linearly independent except for λ in this null set.

The inversion formulae $f(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)$ and $F(\lambda) = \int_S Mf(x) \bar{E}(x, \lambda)dx$, as well as the Parseval equality $[f, g] = (F, G)$, which have been proved for $f \in C^\infty_0(S)$, can be extended to $f \in D_M$ by a standard density argument. They become $f(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)$ and $F(\lambda) = \int_I Mf(x)\bar{E}(x,\lambda)$

dx, with the integrals converging to the functions in the norms of the appropriate Hilbert spaces. These formulae give the expansion of an arbitrary function $f \in D_M$ in eigenfunctions of the differential equation $Lu = \lambda Mu$. This completes the proof of Theorem 1.

Theorem 1 remains valid if L and $\tilde{M}$ are elliptic partial differential operators with sufficiently differentiable coefficients on a domain I in t-dimensional Euclidean space. The only change in the proof is caused by the slightly worse behaviour of the fundamental solution $k(x, y)$ of L. A derivative of order j of $k(x, y)$ is $O(|x - y|^{n-j-t-\epsilon})$ near $x = y$ for any $\epsilon > 0$ (**10**, chap. 3), and thus to ensure that the derivatives of order m are locally square integrable, we must impose the additional condition $n - m > \frac{1}{2}t$ on the orders of L and $\tilde{M}$.

Returning to the case of ordinary differential operators, let $\phi_k(x, \lambda)$ $[k = 1, \ldots, n]$ be a basis of solutions of $Lu = \lambda Mu$, with each ϕ_k analytic in λ for fixed x. For example, we may choose ϕ_k to obey the initial conditions $\phi_k^{(j-1)}(\xi, \lambda) = \delta_{jk}$ $[j, k = 1, \ldots, n]$ for any fixed $\xi \in I$. To prepare for the next section, we now express the eigenfunctions $E_k(x, \lambda)$ in terms of this basis. Since the $E_k(x, \lambda)$ $[k = 1, \ldots, \nu(\lambda)]$ are linearly independent solutions of $Lu = \lambda Mu$, the dimension function $\nu(\lambda) \leq n$. We write

$$E_p(x, \lambda) = \sum_{j=1}^{n} r_{pj}\phi_j(x, \lambda) \qquad [p = 1, \ldots, \nu(\lambda)],$$

where the r_{pj} are complex constants. The Parseval equality

$$||f||^2 = ||F||^2 = \int_R \sum_{p=1}^{\nu(\lambda)} |F_p(\lambda)|^2 d\sigma(\lambda)$$

now takes the form

$$||f||^2 = \int_R \sum_{p=1}^{\nu(\lambda)} \sum_{j,k=1}^{n} (Vf)_j(\lambda)(V\bar{f})_k(\lambda)\bar{r}_{pj}r_{pk}d\sigma(\lambda),$$

where $(Vf)_j(\lambda) = \int_I Mf(x)\bar{\phi}_j(x, \lambda)dx$. We let

$$c_{jk}(\lambda) = \sum_{p=1}^{\nu(\lambda)} \bar{r}_{pj}r_{pk},$$

and then $(c_{jk}(\lambda))$ is a Hermitian positive semi-definite matrix of rank $\nu(\lambda)$. The formulae $d\rho_{jk}(\lambda) = c_{jk}(\lambda)d\sigma(\lambda)$, $\rho_{jk}(0) = 0$, determine an n by n matrix $\rho(\lambda) = (\rho_{jk}(\lambda))$, called a spectral matrix, which is Hermitian, positive semi-definite, and non-decreasing. Let H^* be the Hilbert space of all complex-valued vector functions $F(\lambda) = [F_1(\lambda), \ldots, F_n(\lambda)]$ such that

$$\int_R \sum_{j,k=1}^{n} F_j(\lambda)\bar{F}_k(\lambda)d\rho_{jk}(\lambda) < \infty,$$

with inner product

$$(F, G) = \int_R \sum_{j,k=1}^{n} F_j(\lambda)\bar{G}_k(\lambda)d\rho_{jk}(\lambda).$$

The spectral theorem may be regarded as saying that $(Vf)(\lambda) = \{\int_I Mf(x) \bar{\phi}_j(x, \lambda)dx\}$ defines a unitary mapping V of H onto H^* which diagonalizes A. A straightforward computation gives

$$(V^{-1}F)(x) = \int_R \sum_{j,k=1}^{n} F_j(\lambda)\phi_k(x, \lambda)d\rho_{jk}(\lambda).$$

3. Green's function and the spectral matrix. Let A be a self-adjoint extension of $M^{-1}L$, as in section 2, and let $R_\lambda = (A - \lambda)^{-1}$, for $Im\lambda \neq 0$, be the resolvent of A, a bounded operator in H. Earlier, we made use of the relation $f(x) = \int_S k(x, y)\phi_S(y)Lf(y)dy$ for $x \in S$, where S is a compact subinterval of I and f belongs to $C^n{}_0(S)$. We now modify this in two ways. Instead of using a fundamental solution $k(x, y)$ of L, we use a fundamental solution $k(x, y, \lambda)$ of $L - \lambda M$, and we no longer assume that f vanishes outside a compact subinterval of S. As a result, we have

$$f(x) = \int_S k(x, y, \lambda)\phi_S(y)(L - \lambda M)f(y)dy + u(x)$$

for $x \in S$, where u satisfies $Lu = \lambda Mu$. We apply this relation to $R_\lambda f$ instead of f, obtaining

$$R_\lambda f(x) = \int_S k(x, y, \lambda)\phi_S(y)(L - \lambda M)R_\lambda f(y)dy + u(x).$$

Since $(L - \lambda M)R_\lambda f(y) = Mf(y)$ for any $f \in C^n{}_0(S)$ (which shows also that $R_\lambda f \in C^n(I)$, so that the above relation can be applied to $R_\lambda f$),

$$R_\lambda f(x) = \int_S k(x, y, \lambda)\phi_S(y)Mf(y)dy + u(x).$$

By the Schwarz inequality for the inner product in H, $|R_\lambda f(x) - u(x)| < ||k(x, \ , \lambda)\phi_S(\ )|| \cdot ||f||$, so that $R_\lambda f(x) - u(x)$ is a bounded linear functional of f for $x \in S$. Thus for each $x \in S$ and for $Im\lambda \neq 0$, there exists a function $g(x, \ , \lambda) \in H$ such that

$$R_\lambda f(x) - u(x) = [f, \bar{g}(x, \ , \lambda)] = \int_S g(x, y, \lambda)Mf(y)dy$$

for $f \in C^n{}_0(S)$. Now we can write

$$u(x) = \int_S [k(x, y, \lambda)\phi_S(y) - g(x, y, \lambda)]Mf(y)dy,$$

so that

$$|u(x)| < ||k(x, \ , \lambda)\phi_S(\ ) - g(x, \ , \lambda)|| \cdot ||f||,$$

and $u(x)$ is a bounded linear functional of f for each fixed $x \in S$. Then $u(x) = \int_S v(x, y, \lambda)Mf(y)dy$ for some $v(x, \ , \lambda) \in H$. Since $Lu = \lambda Mu$ for all $f \in C^n(S)$, we can write

$$u(x) = \sum_{i=1}^{n} c_i\phi_i(x, \lambda),$$

where $\phi_i(x, \lambda)$ $(i = 1, \ldots, n)$ form a basis of solutions of $Lu = \lambda Mu$, and the c_i are constants depending on f. This means that $v(x, y, \lambda)$ has the form

$$v(x, y, \lambda) = \sum_{i=1}^{n} v_i(y)\phi_i(x, \lambda).$$

We define

$$G(x, y, \lambda) = g(x, y, \lambda) + \sum_{i=1}^{n} v_i(y)\phi_i(x, \lambda),$$

so that $G(x, \ , \lambda) \in H$ and

$$R_\lambda f(x) = \int_S G(x, y, \lambda)Mf(y)dy = [f, \bar{G}(x, \ , \lambda)].$$

This function G depends on the interval S but is uniquely determined by S. If S' is another compact subinterval which contains S and G' is the corresponding function, it is easy to see that $G(x, y, \lambda) = G'(x, y, \lambda)$ for $x,y \in S$, $Im\lambda \neq 0$. Thus, by taking a sequence of compact subintervals S tending to I, we can extend G uniquely to a function $G(x, y, \lambda)$ defined for $x, y \in I$, $Im\lambda \neq 0$. This function is called the Green's function of A, and has the following properties:

(i) G is analytic in λ for fixed x, y and $Im\lambda \neq 0$, and has continuous partial derivatives with respect to x up to order $n - 2$ on $I \times I$ for fixed λ with $Im\lambda \neq 0$. The partial derivatives of orders $n - 1$ and n are continuous except on $x = y$, and the partial derivative of order $n - 1$ has a jump of $1/p_0(y)$ on $x = y$ if $Im\lambda \neq 0$.

(ii) $G(y, x, \lambda) = \bar{G}(x, y, \bar{\lambda})$.

(iii) Considered as a function of x, G satisfies $Lu = \lambda Mu$ if $x \neq y$.

(iv) G is uniquely determined by A.

(v) If $f \in C^n_0(I)$, then $f(x) = \int_I G(x, y, \lambda)(L - \lambda M)f(y)dy$.

To verify these properties, we begin by noting that for any $f \in D^S$ (the set of functions in D_M such that Mf vanishes identically outside S),

$$\begin{aligned} R_\lambda f(x) &= \int_S \Big[k(x, y, \lambda) + \sum_{i=1}^{n} \phi_i(x, \lambda)v_i(y)\Big]Mf(y)\, dy \\ &= \int_S G(x, y, \lambda)Mf(y)\, dy, \end{aligned}$$

and therefore

$$\int_S \Big[G(x, y, \lambda) - k(x, y, \lambda) - \sum_{i=1}^{n} \phi_i(x, \lambda)v_i(y)\Big]Mf(y)\, dy = 0.$$

This implies, since the functions Mf for f ranging over D^S are dense in $L^2(S)$, that $G(x, y, \lambda)$ has the same differentiability properties with respect to x as

$$k(x, y, \lambda) + \sum_{i=1}^{n} \phi_i(x, \lambda)v_i(y),$$

and thus the same properties as $k(x, y, \lambda)$, since the second term is of class C^n in x. Since $R_\lambda f$ is analytic in λ for $Im\lambda \neq 0$ and all $f \in D_M$ (13), $G(x, y, \lambda)$ is analytic in λ for fixed x, y and $Im\lambda \neq 0$ which completes the proof of (i). To verify (ii), we note that $(R_\lambda f, g) = (f, R_{\bar{\lambda}} g)$ for $f, g \in D_M$, whence

$$\int_I\int_I G(x, y, \lambda)Mf(y)M\bar{g}(x)\, dydx = \int_I\int_I \bar{G}(y, x, \bar{\lambda})Mf(y)M\bar{g}(x)\, dxdy.$$

Since the functions Mf for f ranging over D_M are dense in $L^2(I)$, (ii) follows. Since the difference between two Green's functions would be an eigenfunction of A and the spectrum of A is real, the Green's function of A is unique for $Im\lambda \neq 0$. The property (v) is an immediate consequence of the definition of the resolvent. In view of (i), if $f \in C^n{}_0(S)$, we may apply Green's formula (**4**, p. 86) to L and M separately in (v) to obtain

$$f(x) = \int_S (\bar{L}_y - \overline{\lambda M}_y) G(x, y, \lambda) f(y)\, dy,$$

the subscripts indicating that the differentiations are with respect to y. Then $(L_y - \lambda M_y)\bar{G}(x, y, \lambda) = 0$ for $Im\lambda \neq 0, x \neq y$, and application of (ii) yields property (iii).

Now we express the Green's function in terms of a basis $\phi_k(x, \lambda)$ [$k = 1, \ldots, n$] of solutions of $Lu = \lambda Mu$. It is easily deduced from the above properties of the Green's function that G may be written

$$G(x, y, \lambda) = \sum_{j,k=1}^{n} P^+{}_{jk}(\lambda) \phi_j(x, \lambda) \bar{\phi}_k(y, \bar{\lambda})$$

for $y > x$ and

$$G(x, y, \lambda) = \sum_{j,k=1}^{n} P^-{}_{jk}(\lambda) \phi_j(x, \lambda) \bar{\phi}_k(x, \bar{\lambda})$$

for $y < x$. The $P_{jk}{}^+$ and $P_{jk}{}^-$ are analytic in λ except possibly on the real axis, and $P^-{}_{kj} = \bar{P}^+{}_{jk}$. We define the matrix $P = (P_{jk})$ by $P_{jk}(\lambda) = \frac{1}{2}[P^+{}_{jk}(\lambda) + P^-{}_{jk}(\lambda)]$. Then each P_{jk} is analytic for $Im\lambda \neq 0$ and $\bar{P}_{jk} = P_{kj}$.

THEOREM 2. (Titchmarsh-Kodaira formula.) *The Green's function of A is related to the spectral matrix ρ associated with the basis of solutions $\phi_k(x, \lambda)$ of $Lu = \lambda Mu$ by the formula*

$$P(\mu) = \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{(\lambda - \mu)},$$

where the elements of the matrix P are defined as above, and the formula is to be taken in the sense that

$$P(\mu) - \int_{-N}^{N} \frac{d\rho(\lambda)}{(\lambda - \mu)}$$

is analytic across the real axis on the interval $(-N, N)$.

Proof: Let $f \in D_M$. If $F_j(\lambda) = \int_I Mf(x)\bar{\phi}_j(x, \lambda)dx$, then

$$f(x) = \int_R \sum_{j,k=1}^{n} F_j(\lambda) \phi_k(x, \lambda) d\rho_{jk}(\lambda),$$

as we have seen in the previous section. If

$$u(x) = \int_R \sum_{j,k=1}^n (\lambda - \mu)^{-1} F_j(\lambda) \phi_k(x, \lambda) d\rho_{jk}(\lambda),$$

it is easy to verify that $Lu - \mu Mu = Mf$, or $u = R_\mu f$. Thus

$$\mu(Vu)_j(\lambda) = \mu \int_I Mu(x)\bar{\phi}_j(x, \lambda)\, dx = \int_I Lu(x)\bar{\phi}_j(x, \lambda)\, dx - \int_I Mf(x)\bar{\phi}_j(x, \lambda)\, dx$$

$$= (VAu)_j(\lambda) - (Vf)_j(\lambda) = \lambda(Vu)_j(\lambda) - (Vf)_j(\lambda),$$

or $(\lambda - \mu)(Vu)_j(\lambda) = (Vf)_j(\lambda)$. Now, the Parseval equality applied to u and f yields

$$[u, f] = \int_R \sum_{j,k=1}^n (Vu)_j(\lambda)(V\bar{f})_k(\lambda) d\rho_{jk}(\lambda)$$

$$= \int_R \sum_{j,k=1}^n (\lambda - \mu)^{-1} F_j(\lambda) \bar{F}_k(\lambda) d\rho_{jk}(\lambda),$$

and this is equal to

$$\sum_{j,k=1}^n F_j(\mu)\bar{F}_k(\mu) \int_{-N}^N \frac{d\rho_{jk}(\lambda)}{\lambda - \mu}$$

plus a function which is analytic unless μ is real and $|\mu| > N$. On the other hand, since $u(x) = \int_I G(x, y, \mu) Mf(y) dy$, $[u, f] = \int_I \int_I G(x, y, \mu) M\bar{f}(x) Mf(y) dy dx$. This expression is equal to

$$\sum_{j,k=1}^n P_{jk}(\mu) F_j(\mu) \bar{F}_k(\mu)$$

plus an analytic function. Since f may run through a dense subset of H, which means that F may run through a dense subset of H^*, it follows that the elements of the matrix

$$P(\mu) - \int_{-N}^N \frac{d\rho(\lambda)}{\lambda - \mu}$$

are analytic unless μ is real and $|\mu| > N$.

Another form of the Titchmarsh-Kodaira formula is

$$\rho(\lambda) = \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \frac{1}{2\pi i} \int_\delta^{\lambda + \delta} [P(\mu + i\epsilon) - P(\mu - i\epsilon)]\, d\mu,$$

with ρ normalized to be continuous from the right and $\rho(0) = 0$, and with the formula interpreted as in Theorem 2 .We write

$$P(\mu + i\epsilon) - P(\mu - i\epsilon) = \int_{-N}^N \left[\frac{1}{\tau - \mu - i\epsilon} - \frac{1}{\tau - \mu + i\epsilon}\right] d\rho(\tau) + w(\mu)$$

$$= 2i \int_{-N}^N \frac{\epsilon\, d\rho(\tau)}{(\tau - \mu)^2 + \epsilon^2} + w(\mu),$$

where $w(\mu)$ is analytic unless μ is real and $|\mu| > N$. Then

$$\frac{1}{2i}\lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\int_\delta^{\lambda+\delta}[P(\mu+i\epsilon)-P(\mu-i\epsilon)]\,d\mu$$
$$=\lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\int_\delta^{\lambda+\delta}\int_{-N}^{N}\frac{\epsilon d\rho(\tau)d\mu}{(\tau-\mu)^2+\epsilon^2}+w_1(\lambda)$$
$$=\lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\int_{-N}^{N}\left[\int_\delta^{\lambda+\delta}\frac{\epsilon d\mu}{(\tau-\mu)^2+\epsilon^2}\right]d\rho(\tau)+w_1(\lambda)$$
$$=\lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\int_{-N}^{N}\left[\tan^{-1}\left(\frac{\lambda+\delta-\tau}{\epsilon}\right)-\tan^{-1}\left(\frac{\delta-\tau}{\epsilon}\right)\right]d\rho(\tau)+w_1(\lambda)$$
$$=\pi\lim_{\delta\to 0+}[\rho(\lambda+\delta)-\rho(\delta)]+w_1(\lambda)$$
$$=\pi\rho(\lambda)+w_1(\lambda),$$

where $w_1(\lambda)$ is analytic unless λ is real and $|\lambda| > N$, as desired.

4. Boundary conditions. Let D be the set of functions f in H with continuous derivatives up to order $n-1$ on I, such that $f^{(n-1)}$ is absolutely continuous on every compact subinterval of I, so that $f^{(n)}$ exists almost everywhere on I, and such that Lf belongs to $L^2(I)$. Let T be the operator in H with domain D defined by $Tf = M^{-1}Lf$ for $f \in D$. We assume that T has at least one self-adjoint restriction. Let R_λ be the resolvent of some self-adjoint restriction of T, so that $R_\lambda f(x) = \int_I G(x, y, \lambda)Mf(y)dy$ for $f \in D_M$, $Im\lambda \neq 0$. Then R_λ is a bounded operator for $Im\lambda \neq 0$, whose adjoint R^*_λ is $R_{\bar\lambda}$. Let $\mathfrak{E}(\lambda)$ be the eigenspace of T corresponding to the value λ, the set of all solutions in D of the differential equation $Lu = \lambda Mu$.

LEMMA 1: *T is a closed operator whose domain consists of all $f \in H$ of the form $f = R_\lambda h + w$, where $h \in H$, $w \in \mathfrak{E}(\lambda)$, $Im\lambda \neq 0$.*

Proof: Since R_λ maps H into D and $\mathfrak{E}(\lambda)$ is contained in D, it is clear that every f of this form belongs to D. Conversely, suppose $f \in D$ is given. Let $h = Tf - \lambda f$, $w = f - R_\lambda h$. Then $Tw = Tf - TR_\lambda h = Tf - \lambda R_\lambda h - h = Tf - h - \lambda(h - w) = \lambda w$, and thus $w \in \mathfrak{E}(\lambda)$, while $f = R_\lambda h + w$. When f is written in this way, $Tf - \lambda f = h$. To show that T is a closed operator, take a sequence f_k in D such that

$$f = \lim_{k\to\infty} f_k \quad\text{and}\quad f^* = \lim_{k\to\infty} Tf_k$$

exist. We can write $f_k = R_\lambda(Tf_k - \lambda f_k) + w_k$, and we deduce that

$$w = \lim_{k\to\infty} w_k$$

exists and belongs to $\mathfrak{E}(\lambda)$. Letting $k \to \infty$, we obtain $f = R_\lambda(f^* - \lambda f) + w$, which implies $f \in D$ and $Tf = f^*$. This proves that T is closed.

Since T is closed and its domain D is dense in H, T has a closed adjoint T^* whose domain D^* is dense in H. Also, $T = T^{**} = (T^*)^*$. If $\mathfrak{M}$ is a subspace of H, let $H - \mathfrak{M}$ denote the orthogonal complement of $\mathfrak{M}$ in H.

LEMMA 2: *D^* consists of all $g \in D$ of the form $g = \bar{R}_\lambda z$, where $z \in H - \mathfrak{E}(\bar{\lambda})$. The adjoint operator T^* is a restriction of T and is closed and symmetric.*

Proof. $g^* = T^*g$ means $[Tf, g] = [f, g^*]$ for every $f \in D$. By Lemma 1, any $f \in D$ may be written $f = R_{\bar{\lambda}}h + w$, with $h \in H, w \in \mathfrak{E}(\bar{\lambda})$, and then $Tf = \bar{\lambda}f + h$. Substituting in the equation $[Tf, g] = [f, g^*]$, we obtain $[\bar{\lambda}f + h, g] = [R_{\bar{\lambda}}h + w, g^*]$, or $[\bar{\lambda}R_{\bar{\lambda}}h + \bar{\lambda}w + h, g] = [R_{\bar{\lambda}}h + w, g^*]$. This is equivalent to

$$[h, \lambda R_{\bar{\lambda}}^*g + g - R_{\bar{\lambda}}^*g^*] + [w, \lambda g - g^*] = 0$$

for all $h \in H$, $w \in \mathfrak{E}(\bar{\lambda})$. Then $\lambda g - g^* = z$ is orthogonal to $\mathfrak{E}(\bar{\lambda})$ and $g = R_{\bar{\lambda}}^*z = \bar{R}_\lambda z$. Since D and D^* are obviously unaltered by complex conjugation, $\bar{g} = R_\lambda \bar{z}$ belongs to D^*, and by Lemma 1, $\bar{g} \in D$, so that $g \in D$. Conversely, if $g = \bar{R}_\lambda z$ with $z \in \mathfrak{E}(\bar{\lambda})$, we let $g^* = -z - \lambda g$, and find that $g \in D^*$, $g^* = T^*g$. Thus D^* is as described in the statement of the lemma, and is contained in D. Now $T \supseteq T^*$, $T = (T^*)^* \supseteq T^*$, and T^* is symmetric.

Since $T = (T^*)^*$, the theory of the Cayley transform implies that $D = D^* \oplus \mathfrak{E}(i) \oplus \mathfrak{E}(-i)$, where $\oplus$ denotes a direct sum. Let the linear spaces $\mathfrak{E}(i)$ and $\mathfrak{E}(-i)$ have dimensions τ^+ and τ^- respectively. Since the set of solutions of the differential equation $Lu = \lambda Mu$ is a linear space of dimension n, neither of these dimensions can exceed n. The defect index of T^* is (τ^+, τ^-), and T^* has self-adjoint extensions, which are restrictions of T, if and only if $\tau^+ = \tau^-$. We have already assumed that T has at least one self-adjoint restriction. This assumption is equivalent to the assumption $\tau^+ = \tau^- = \tau$, which we now make .We will characterize all the self-adjoint extensions of T^* by boundary conditions. The self-adjoint extensions of T^* are in one to one correspondence with the unitary operators U of $\mathfrak{E}(i)$ onto $\mathfrak{E}(-i)$. Corresponding to any such U there is a self-adjoint extension A of T^* whose domain D_A is the set of all $f \in D$ of the form $f = f^* + (1 - U)f^+$ with $f^* \in D^*, f^+ \in \mathfrak{E}(i)$, where 1 is the identity operator on $\mathfrak{E}(i)$. Conversely, every such A has a domain of this type. Let $y_1, \ldots, y_\tau$ and $z_1, \ldots, z_\tau$ be orthonormal bases for $\mathfrak{E}(i)$ and $\mathfrak{E}(-i)$ respectively. Then every $f^+ \in \mathfrak{E}(i)$ is of the form $f^+ = \Sigma_{j=1}^\tau a_j y_j$, for some constants a_j. The effect of U on f^+ can be represented by a unitary matrix $U = (u_{jk})$;

$$Uf^+ = \sum_{j=1}^{\tau} a_j U y_j = \sum_{j=1}^{\tau} a_j \sum_{k=1}^{\tau} u_{jk} z_k.$$

Thus $f \in D_A$ if and only if

$$f = f^* + \sum_{j=1}^{\tau} a_j g_j,$$

where $f^* \in D^*$ and

$$g_j = y_j - \sum_{k=1}^{\tau} u_{jk} z_k, \qquad [j = 1, \ldots, \tau].$$

Green's formula (**4**, p. 86) is

$$\int_\alpha^\beta [Lf(x)\bar{g}(x) - f(x)L\bar{g}(x)]dx = [fg](\beta) - [fg](\alpha),$$

for $f, g \in D$, where $[\alpha, \beta]$ is any compact subinterval of I. Here $[fg](x)$ is a bilinear form in the derivatives up to order $(n-1)$ of f and g which is non-degenerate for all $x \in I$. It follows easily from Green's formula that $[fg](x)$ is skew-Hermitian, $[fg](x) = -[\bar{g}\bar{f}](x)$. If I is the interval (a, b), then

$$[fg](a) = \lim_{x \to a+0} [fg](x) \quad \text{and} \quad [fg](b) = \lim_{x \to b-0} [fg](x)$$

exist for all $f, g \in D$. We let

$$< fg > = [fg](b) - [fg](a).$$

A homogeneous boundary condition is a condition on $f \in D$ of the form $< f\alpha > = 0$, where α is a fixed function in D. The conditions $< f\alpha_j > = 0$, $[j = 1, \ldots, p]$ are said to be linearly independent if the only set of complex numbers $\gamma_1, \ldots, \gamma_p$ for which

$$\sum_{j=1}^{p} \gamma_j < f\alpha_j > = 0$$

identically in $f \in D$ is $\gamma_1 = \ldots = \gamma_p = 0$. It is easily seen, since $[Tf, g] - [f, T^*g] = < fg > = 0$ for all $f \in D$, $g \in D^*$, that these boundary conditions are linearly independent if and only if the functions $\alpha_1, \ldots, \alpha_p$ are linearly independent (mod D^*). A set of p linearly independent boundary conditions $< f\alpha_j > = 0$ $[j = 1, \ldots, p]$ is said to be self-adjoint if $< \alpha_j\alpha_k > = 0$ for $j, k = 1, \ldots, p$. Two sets of boundary conditions are said to be equivalent if the sets of functions satisfying the two sets of conditions are identical.

THEOREM 3: *If A is a self-adjoint extension of T^* with domain D_A, then there exists a self-adjoint set of τ linearly independent boundary conditions such that D_A is the set of all $f \in D$ satisfying these conditions. Conversely, corresponding to a self-adjoint set of τ linearly independent boundary conditions, there exists a self-adjoint extension of T^* whose domain D_A is the set of all $f \in D$ satisfying these boundary conditions.*

Proof. The proof is very similar to the proof of Theorem 3 in (**3**). First, suppose that A is a self-adjoint extension of T^* with domain D_A. There exists a unitary matrix $U = (u_{jk})$ such that

$$f = f^* + \sum_{j=1}^{\tau} a_j g_j$$

for all $f \in D_A$, where $f^* \in D^*$ and

$$g_j = y_j - \sum_{k=1}^{\tau} u_{jk} z_k, \qquad [j = 1, \ldots, \tau].$$

We will show that the conditions $< fg_j > = 0$ $[j = 1, \ldots, \tau]$ are self-adjoint and that $f \in D_A$ if and only if f satisfies these conditions. It is clear that $g_j \in \mathfrak{E}(i) \oplus \mathfrak{E}(-i)$, and thus

$$\sum_{j=1}^{\tau} \gamma_j g_j \in D^* \quad \text{implies} \quad \sum_{j=1}^{\tau} \gamma_j g_j = 0.$$

Then

$$\sum_{j=1}^{\tau} \gamma_j y_j = 0 \quad \text{and} \quad \sum_{j=1}^{\tau} \gamma_j \sum_{k=1}^{\tau} u_{jk} z_k = 0,$$

and since the y_j are linearly independent, this can happen only if $\gamma_1 = \ldots = \gamma_\tau = 0$. It follows that the boundary conditions $< fg_j > = 0$ $[j = 1, \ldots, \tau]$ are linearly independent. It follows from Green's formula that $< y_j y_k > = 2i\delta_{jk}$, $< z_j z_k > = -2i\delta_{jk}$, $< y_j z_k > = 0$ for $j, k = 1, \ldots, \tau$, and thus

$$\begin{aligned} < g_j g_k > &= < y_j y_k > - \sum_{q=1}^{\tau} u_{jq} < z_q y_k > - \sum_{p=1}^{\tau} \bar{u}_{kp} < y_j z_p > \\ &\quad + \sum_{p,q=1}^{\tau} u_{jq} \bar{u}_{kp} < z_q z_p > \\ &= 2i\delta_{jk} - 2i \sum_{q=1}^{\tau} u_{jq} \bar{u}_{kq}. \end{aligned}$$

This vanishes for $j, k = 1, \ldots, \tau$ because U is a unitary matrix. Thus the boundary conditions are self-adjoint. If $f \in D_A$,

$$f = f^* + \sum_{p=1}^{\tau} a_p g_p, \quad \text{and} \quad < fg_j > = < f^* g_j > + \sum_{p=1}^{\tau} a_p < g_p g_j >.$$

The first term vanishes since $f^* \in D^*$, and the second term vanishes because the boundary conditions are self-adjoint. Thus if $f \in D_A$, f satisfies the boundary conditions. Conversely, suppose $f \in D$ and $< fg_j > = 0$ $[j = 1, \ldots, \tau]$. We can write

$$f = f^* + \sum_{j=1}^{\tau} b_j y_j + \sum_{j=1}^{\tau} c_j z_j$$

for some constants b_j, c_j. Then $< fg_j > = 0$ implies

$$b_j = - \sum_{k=1}^{\tau} \bar{u}_{jk} c_k,$$

or, equivalently,

$$c_p = - \sum_{j=1}^{\tau} u_{jp} b_j,$$

and this yields

$$\begin{aligned} f &= f^* + \sum_{j=1}^{\tau} b_j (y_j - \sum_{k=1}^{\tau} u_{jk} z_k) \\ &= f^* + \sum_{j=1}^{\tau} b_j g_j \in D_A. \end{aligned}$$

To prove the converse, we assume that $< fg_j > = 0$ $[j = 1, \ldots, \tau]$ is a self-adjoint set of τ linearly independent boundary conditions. It suffices to show that there exists a unitary matrix $U = (u_{jk})$ such that the given boundary conditions are equivalent to a self-adjoint set of the form $<f\tilde{g}_j> = 0$, where

$$\tilde{g}_j = y_j - \sum_{k=1}^{\tau} u_{jk}z_k, \qquad [j = 1, \ldots, \tau].$$

Since $g_j \in D$, there exists a unique set of complex numbers b_{jk}, c_{jk} such that

$$g_j = g^*_j + \sum_{k=1}^{\tau} b_{jk}y_k + \sum_{k=1}^{\tau} c_{jk}z_k,$$

with $g^*_j \in D^*$ $[j = 1, \ldots, \tau]$. The relations

$$< g_jg_k > = 0 \quad \text{imply} \quad \sum_{p=1}^{\tau} (b_{jp}\bar{b}_{kp} - c_{jp}\bar{c}_{kp}) = 0.$$

If the matrices B and C are defined by $B = (b_{jk})$, $C = (c_{jk})$, this may be written $BB^* = CC^*$. The linear independence of the $g_j(\text{mod } D^*)$ is equivalent to the fact that the two sets of functions

$$\tilde{y}_j = \sum_{k=1}^{\tau} b_{jk}y_k, \quad \tilde{z}_j = \sum_{k=1}^{\tau} c_{jk}z_k \qquad [j = 1, \ldots, \tau]$$

are each linearly independent. Suppose there exist constants $\gamma_1, \ldots, \gamma_\tau$, not all zero, such that

$$y = \sum_{j=1}^{\tau} \gamma_j\tilde{y}_j = 0.$$

If

$$z = \sum_{j=1}^{\tau} \gamma_j\tilde{z}_j,$$

then $< zz > = 0$. Since $z \in \mathfrak{E}(-i)$, $[Tz, z] - [z, Tz] = -2i[z, z] = <zz> = 0$, and thus $z = 0$. Now $y = z = 0$ implies that there exist constants $\gamma_1, \ldots, \gamma_\tau$, not all zero, such that

$$\sum_{j=1}^{\tau} \gamma_jg_j = \sum_{j=1}^{\tau} \gamma_jg^*_j + \sum_{j=1}^{\tau} \gamma_j\tilde{y}_j + \sum_{j=1}^{\tau} \gamma_j\tilde{z}_j = 0(\text{mod } D^*),$$

which contradicts the linear independence (mod D^*) of the g_j. Thus the $\tilde{y}_j$ and $\tilde{z}_j$ are linearly independent, which implies that the matrices B and C are non-singular. Let $U = -B^{-1}C$. Then $UU^* = B^{-1}CC^*B^{*-1} = B^{-1}BB^*B^{*-1}$, which is the identity matrix, and thus U is unitary. Now we define

$$\tilde{g}_j = y_j - \sum_{k=1}^{\tau} u_{jk}z_k, \qquad [j = 1, \ldots, \tau].$$

If $B^{-1} = (d_{jk})$, then

$$\sum_{j=1}^{\tau} d_{pj}g_j = \sum_{j=1}^{\tau} d_{pj}g^*_j + y_p - \sum_{k=1}^{\tau} u_{pk}z_k,$$

and

$$\sum_{j=1}^{\tau} \bar{d}_{jp} < fg_j > = < f\bar{g}_p > \qquad f \in D, p = 1, \ldots, \tau,$$

Thus f satisfies $< fg_j > = 0$ $[j = 1, \ldots, \tau]$ if and only if $< f\bar{g}_j > = 0$ $[j = 1, \ldots, \tau]$, and the theorem is proved.

If I is a finite interval $[a, b]$, the author has shown in **(1)** that n boundary conditions are required. It is also shown in **(1)** that if M has real coefficients, so that m is an even integer, $m = 2s$, and these conditions include $f(a) = \ldots = f^{(s-1)}(a) = f(b) = \ldots = f^{(s-1)}(b) = 0$, then H is a space of functions satisfying these $m = 2s$ boundary conditions, and only $(n - m)$ conditions are actually involved in defining the self-adjoint extensions. It may be conjectured that, in general, if I is a finite interval, the definition of H involves m boundary conditions, and $\tau = n - m$. It is not clear how many boundary conditions, if any, are needed to define H if I is an infinite interval.

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A SINGULAR BOUNDARY VALUE PROBLEM FOR A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR

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If $(x^2+1)^{\frac{1}{2}}g(x) \in L^1(-\infty, \infty)$ the differential expression $l(y) = -y'' + g(x)y$ generates a closed operator L on $L^2(-\infty, \infty)$, with domain D consisting of those functions $y \in L^2$ with absolutely continuous derivatives and such that $l(y) \in L^2$. The case where $g(x)$ is real-valued has been extensively investigated and yields an expansion of any $f \in L^2$ in terms of the characteristic functions of L. We shall investigate the case where g is complex-valued.

We shall find that there is a function $W(s)$, analytic for $Im\, s > 0$ and continuous for $Im\, s \geqslant 0$, such that the squares of its zeros in $Im\, s > 0$ constitute a bounded set which is the point spectrum of L. The continuous spectrum of L is the set of $\lambda \geqslant 0$. In proving an expansion theorem real zeros of W cause difficulties and it is necessary to assume (Case II) that W has only a finite number of zeros in $Im\, s \geqslant 0$. The simplest form of the expansion is obtained if W has no real zeros except possibly at $s = 0$, and this must be a simple zero (Case I).

Naimark **(2)** has considered the same differential operator on $[0, \infty)$ with a boundary condition at 0 and obtains similar results. He uses a modification of a technique for singular self-adjoint problems (**1**, chap. 9), while we shall use a modification of the Cauchy Integral technique used for non-singular non self-adjoint problems (**1**, chap. 12) and for general self-adjoint problems **(3)**.

In § 1 we investigate the properties of certain solutions of $l(y) = \lambda y$ and introduce $W(s)$. We construct the Green's function and investigate the spectrum of L in § 2. An expansion of the Green's function for the general case is given in § 3, while in § 4 and § 5 we deal with Cases I and II respectively.

1. Solutions of $l(y) = \lambda y$. We shall set $\lambda = s^2$ and denote $Re\, s$ by σ and $Im\, s$ by τ. Also $\lambda^{\frac{1}{2}}$ will denote the root of λ with $0 \leqslant \arg\lambda^{\frac{1}{2}} < \pi$ and K will denote any constant whose value is unimportant.

It is easily seen by using variation of constants that a solution of $l(y) = s^2y$ will satisfy an integral equation of the form

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$$y(x) = c_1 e^{isx} + c_2 e^{-isx} \tag{1.1}$$
$$+ \frac{e^{isx}}{2is}\int_{x_1}^{x} e^{-is\xi} g(\xi) y(\xi)\, d\xi - \frac{e^{-isx}}{2is}\int_{x_2}^{x} e^{is\xi} g(\xi) y(\xi)\, d\xi,$$

and conversely. In estimating solutions of (1.1) we shall make frequent use of the following lemma, which we state without proof.

LEMMA 1.1. *If ϕ and ψ are piecewise continuous functions on $[a, b]$ and χ is integrable and non-negative on $[a, b]$ then*

$$\phi(x) \leqslant \psi(x) + \int_a^x \chi(\xi)\phi(\xi)\, d\xi, \qquad x \in [a, b],$$

implies

$$\phi(x) \leqslant \psi(x) + \int_a^x \chi(\xi)\psi(\xi) \exp\left[\int_\xi^x \chi(u) du\right] d\xi, \qquad x \in [a, b],$$

and

$$\phi(x) \leqslant \psi(x) + \int_x^b \chi(\xi)\phi(\xi)\, d\xi, \qquad x \in [a, b],$$

implies

$$\phi(x) \leqslant \psi(x) + \int_x^b \chi(\xi)\psi(\xi) \exp\left[\int_x^\xi \chi(u) du\right] d\xi, \qquad x \in [a, b].$$

LEMMA 1.2. *The solutions $y(x, s)$ of* (1.1) *with $c_1 = -c_2 = 1/2is$, $x_1 = x_2 = 0$, and $\bar{y}(x, s)$ of* (1.1) *with $c_1 = c_2 = \frac{1}{2}$, $x_1 = x_2 = 0$ exist for all x and s, and for any fixed x are entire, and even functions of s.*

As this result follows from well-known theorems we omit the proof. We note that y and $\bar{y}$ satisfy the initial conditions

$$\begin{aligned} y(0, s) &= 0 \qquad & y'(0, s) &= 1 \\ \bar{y}(0, s) &= 1 \qquad & \bar{y}'(0, s) &= 0 \end{aligned} \tag{1.2}$$

and that a modification of Lemma 1.1 yields the following estimates:

$$|y(x, s)| \leqslant \frac{K|x|e^{|\tau x|}}{1 + |sx|}, \quad |\bar{y}(x, s)| \leqslant \left(1 + \frac{K|x|}{1 + |sx|}\right)e^{|\tau x|}. \tag{1.3}$$

LEMMA 1.3. *The solutions $y_1(x, s)$ of* (1.1) *with $c_1 = 1$, $c_2 = 0$, $x_1 = x_2 = \infty$, and $y_2(x, s)$ of* (1.1) *with $c_1 = 0$, $c_2 = 1$, $x_1 = x_2 = -\infty$ exist for all x and for $\tau \geqslant 0$. For any fixed x they are continuous in s for $\tau \geqslant 0$ and analytic in s for $\tau > 0$.*

Proof. We shall prove the result for $y_1(x, s)$ only as the proof for $y_2(x, s)$ is similar. Setting $\phi_0(x, s) = 0$ and

$$\phi_{n+1}(x, s) = e^{isx} - \int_x^\infty \frac{\sin s(x - \xi)}{s} g(\xi)\phi_n(\xi, s)\, d\xi$$

and using the inequality

$$\frac{|\sin sx|}{|s|} \leqslant K|x|e^{|\tau x|}(1 + |sx|)^{-1}$$

we see that the successive approximations exist for all x and for $\tau \geqslant 0$. For fixed x, $\phi_n(x, s)$ is continuous in s for $\tau \geqslant 0$ and analytic in s for $\tau > 0$. An induction yields

$$|\phi_{n+1}(x, s) - \phi_n(x, s)| < \left[K \int_x^\infty \xi |g(\xi)| d\xi\right]^n e^{-\tau x}/n!$$

for $x \geqslant 0$. This implies the uniform convergence of the successive approximations for $x \geqslant 0$, $\tau \geqslant 0$. An application of Lemma 1.1 proves the uniqueness and when we define

$$y_1(x, s) = y_1'(0, s) y(x, s) + y_1(0, s) \bar{y}(x, s)$$

for $x < 0$ the regularity follows from the uniform convergence for $x \geqslant 0$ and from the known properties of $y(x, s)$ and $\bar{y}(x, s)$. The fact that the integral equation is also satisfied for $x < 0$ follows from a few manipulations with the definition of $y_1(x, s)$ for $x < 0$.

Applying Lemma 1.1 yields estimates on $y_1(x, s)$ and $y_2(x, s)$ which allow us to draw certain conclusions about the asymptotic behaviour of these two solutions:

$$|y_1(x, s)| < \exp\left[-\tau x + K \int_x^\infty \xi |g(\xi)| d\xi\right], \qquad x \geqslant 0, \tag{1.4}$$

$$|y_1(x, s)| < \exp\left[-\tau x + \frac{1}{|s|} \int_x^\infty |g(\xi)| d\xi\right], \qquad s \neq 0, \tag{1.5}$$

$$|y_2(x, s)| < \exp\left[\tau x + K \int_{-\infty}^x |\xi g(\xi)| d\xi\right], \qquad x \leqslant 0, \tag{1.6}$$

$$|y_2(x, s)| < \exp\left[\tau x + \frac{1}{|s|} \int_{-\infty}^x |g(\xi)| d\xi\right], \qquad s \neq 0. \tag{1.7}$$

Lemma 1.4. *The solutions $y_1(x, s)$ and $y_2(x, s)$ have the following asymptotic behaviour:*

$$y_1(x, s) = e^{isx}(1 + o(1)), \quad y_1'(x, s) = e^{isx}(is + o(1)) \qquad \text{as } x \to \infty. \tag{1.8}$$

$$y_1(x, s) = e^{isx}\left(1 + O\left(\frac{1}{s}\right)\right), \quad y_1'(x, s) = ise^{isx}\left(1 + O\left(\frac{1}{s}\right)\right) \quad \text{as } |s| \to \infty. \tag{1.9}$$

$$y_2(x, s) = e^{-isx}(1 + o(1)), \quad y_2'(x, s) = e^{-isx}(-is + o(1)) \quad \text{as } x \to -\infty. \tag{1.10}$$

$$y_2(x, s) = e^{-isx}\left(1 + O\left(\frac{1}{s}\right)\right), \quad y_2'(x, s) = -ise^{-isx}\left(1 + O\left(\frac{1}{s}\right)\right) \quad \text{as } |s| \to \infty. \tag{1.11}$$

Formulas (1.8) *and* (1.10) *hold uniformly in s for $\tau \geqslant 0$ and* (1.9) *and* (1.11) *hold uniformly in x, for all x.*

Proof. For (1.8) and (1.10) we use (1.4) and (1.6) respectively in the appropriate form of (1.1) and its derivative. For (1.9) and (1.11) we use (1.5) and (1.7) respectively in the same way.

LEMMA 1.5. *The Wronskian* $W(s) = y_1y_2' - y_1'y_2$ *is not identically zero, and can be calculated from the formulas*

$$W(s) = -2is + \int_{-\infty}^{\infty} e^{-isx}g(x)y_1(x, s)\,dx$$
$$= -2is + \int_{-\infty}^{\infty} e^{isx}g(x)y_2(x, s)\,dx.$$

Proof. It follows immediately from the regularity properties of y_1 and y_2 that $W(s)$, which is independent of x, is continuous in s for $\tau \geqslant 0$ and analytic in s for $\tau > 0$. Direct computation with the integral equations defining y_1 and y_2 yields

$$W(s) = -2is + \int_{-\infty}^{x} e^{is\xi}g(\xi)y_2(\xi, s)\,d\xi + \int_{x}^{\infty} e^{-is\xi}g(\xi)y_1(\xi, s)\,d\xi + R \tag{1.12}$$

where

$$|R| = \left| -\int_x^{\infty}\frac{\sin s(x-\xi)}{s}g(\xi)y_1(\xi, s)\,d\xi\cdot\int_{-\infty}^{x}\cos s(x-\xi)g(\xi)y_2(\xi, s)\,d\xi + \int_x^{\infty}\cos s(x-\xi)g(\xi)y_1(\xi, s)\,d\xi\cdot\int_{-\infty}^{x}\frac{\sin s(x-\xi)}{s}g(\xi)y_2(\xi, s)\,d\xi\right|$$
$$< \frac{2}{|s|}\left[\int_x^{\infty}|g(\xi)|\exp\left(\frac{1}{|s|}\int_{\xi}^{\infty}|g(u)|du\right)d\xi\right]\cdot\left[\int_{-\infty}^{x}|g(\xi)|\exp\left(\frac{1}{|s|}\int_{-\infty}^{\xi}|g(u)|du\right)d\xi\right]$$
$$= 2|s|\left[\exp\left(\frac{1}{|s|}\int_x^{\infty}|g(u)|du\right) - 1\right]\cdot\left[\exp\left(\frac{1}{|s|}\int_{-\infty}^{x}|g(u)|du\right) - 1\right].$$

Here (1.5) and (1.7) have been used and it now follows that for $s \neq 0$

$$\lim_{x\to\infty} R = \lim_{x\to-\infty} R = 0.$$

Using (1.4) and (1.6) we see that $|y_j(0, s)| < K$ and $|y_j'(0, s)| < K + |s|$ for $j = 1, 2$ and thus using (1.3) we obtain the further estimates

$$|e^{-isx}y_1(x, s)| < K(1 + |x|), \quad |e^{isx}y_2(x, s)| < K(1 + |x|)$$

for all x and $\tau \geqslant 0$. Thus

$$\int_{-\infty}^{\infty} e^{-isx}g(x)y_1(x, s)\,dx$$

and

$$\int_{-\infty}^{\infty} e^{+isx}g(x)y_2(x, s)\,dx$$

converge uniformly in s for $\tau \geqslant 0$ and thus are continuous functions of s. Using this fact and the results about R obtained above it follows that we

may take the limit of (1.12) as $x \to \infty$ or as $x \to -\infty$ and obtain the desired formulas for $s \neq 0$. The result follows for $s = 0$ by continuity.

To see that $W(s)$ is not identically zero we note that

$$
\begin{aligned}
|W(s)| &= \left| -2is + \int_{-\infty}^{\infty} e^{-isx} g(x) y_1(x, s)\, dx \right| \\
&\geq 2|s| - \int_{-\infty}^{\infty} |g(x)| \exp\left(\frac{1}{|s|} \int_x^{\infty} |g(u)|\, du \right) dx \\
&= |s| \left[3 - \exp\left(\frac{1}{|s|} \int_{-\infty}^{\infty} |g(u)|\, du \right) \right].
\end{aligned}
$$

Thus for large $|s|$, $W(s)$ cannot be zero.

COROLLARY 1.1. *The zeros of $W(s)$ form an at most countable, bounded subset of $\tau > 0$, with limit points only on the real axis $\tau = 0$.*

This follows immediately from the analyticity of $W(s)$ in $\tau > 0$ and the fact that $W(s) = 0$ implies

$$
|s| < \int_{-\infty}^{\infty} |g(x)|\, dx.
$$

We now complete our discussion of the asymptotic behaviour of $y_1(x, s)$ and $y_2(x, s)$.

LEMMA 1.6.

$$
y_1(x, s) = e^{isx}\left(-\frac{W(s)}{2is} + o(1) \right) \qquad \text{as } x \to -\infty \tag{1.13}
$$

$$
y_2(x, s) = e^{-isx}\left(-\frac{W(s)}{2is} + o(1) \right) \qquad \text{as } x \to +\infty \tag{1.14}
$$

uniformly in s for $\tau \geq \delta > 0$.

Proof. We shall prove only (1.14) as (1.13) is similar.

$$
\begin{aligned}
y_2(x, s) &= e^{-isx} + \int_{-\infty}^{x} \frac{\sin s(x - \xi)}{s} g(\xi) y_2(\xi, s)\, d\xi \\
&= e^{-isx}\left[1 + \int_{-\infty}^{x} e^{+isx} \frac{\sin s(x - \xi)}{s} g(\xi) y_2(\xi, s)\, d\xi \right] \\
&= e^{-isx}\left[1 - \int_{-\infty}^{x} \frac{e^{is\xi}}{2is} g(\xi) y_2(\xi, s)\, d\xi \right. \\
&\qquad \left. + \int_{-\infty}^{x} \frac{e^{2is(x-\xi)}}{2is} g(\xi) e^{is\xi} y_2(\xi, s)\, d\xi \right] \\
&= e^{-isx}\left[-\frac{W(s)}{2is} + \int_{x}^{\infty} \frac{e^{is\xi}}{2is} g(\xi) y_2(\xi, s)\, d\xi \right. \\
&\qquad \left. + \int_{-\infty}^{x} \frac{e^{2is(x-\xi)}}{2is} g(\xi) e^{is\xi} y_2(\xi, s)\, d\xi \right].
\end{aligned}
$$

Now, the integral over the range from x to ∞ approaches zero as $x \to \infty$ uniformly in s for $\tau \geq \delta > 0$ from the proof of Lemma 1.5, and for $x \geq 0$

$$\left| \int_{-\infty}^{x} \frac{e^{2is(x-\xi)}}{2is} g(\xi) e^{is\xi} y_2(\xi, s)\, d\xi \right|$$
$$< \frac{1}{2\delta} \int_{-\infty}^{x} e^{-2\delta(x-\xi)} |g(\xi)| \exp\left[\frac{1}{\delta} \int_{-\infty}^{\xi} |g(u)|\, du \right] d\xi$$
$$< K \int_{-\infty}^{x/2} e^{-2\delta(x-\xi)} |g(\xi)|\, d\xi + K \int_{x/2}^{x} e^{-2\delta(x-\xi)} |g(\xi)|\, d\xi$$
$$< K e^{-\delta x} \int_{-\infty}^{x/2} |g(\xi)|\, d\xi + K \int_{x/2}^{x} |g(\xi)|\, d\xi,$$

which certainly approaches 0 as $x \to \infty$.

2. The Green's Function. If $W(s) \neq 0$ we may define

$$K(x, \xi, s) = \frac{1}{W(s)} \begin{cases} y_1(x, s) y_2(\xi, s) & \xi < x \\ y_1(\xi, s) y_2(x, s) & x < \xi. \end{cases}$$

THEOREM 2.1. *If λ is not real and non-negative, and $W(\lambda^{\frac{1}{2}}) \neq 0$ then the Green's function $G(x, \xi, \lambda)$ for $l(y) - \lambda y = f$ on the interval $-\infty < x < \infty$ is $K(x, \xi, \lambda^{\frac{1}{2}})$, that is, if $f \in L^2(-\infty, \infty)$ then $y = \int_{-\infty}^{\infty} K(x, \xi, \lambda^{\frac{1}{2}}) f(\xi) d\xi$ belongs to D and $l(y) - \lambda y = f$ almost everywhere.*

Proof. Let $\lambda^{\frac{1}{2}} = s = \sigma + i\tau$ as usual. By assumption $\tau > 0$ and $W(s) \neq 0$ so $y_1(x, s)$ and $y_2(x, s)$ are linearly independent solutions of $l(y) = \lambda y$. It follows from variation of constants that the general solution of $l(y) - \lambda y = f$ is

$$y(x) = c_1 y_1(x, s) + c_2 y_2(x, s)$$
$$+ \frac{y_1(x, s)}{W(s)} \int_{-\infty}^{x} f(\xi) y_2(\xi, s)\, d\xi + \frac{y_2(x, s)}{W(s)} \int_{x}^{\infty} f(\xi) y_1(\xi, s)\, d\xi$$
$$= c_1 y_1(x, s) + c_2 y_2(x, s) + \int_{-\infty}^{\infty} K(x, \xi, s) f(\xi)\, d\xi,$$

where the existence of the integrals is trivial. As $|K(x, \xi, s)| < K \exp[-\tau|x-\xi|]$, $y(x) = \int_{-\infty}^{\infty} K(x, \xi, s) f(\xi) d\xi$ is bounded by the convolution of a function in L^1 and a function in L^2. Thus $y \in L^2(-\infty, \infty)$. As $W(s) \neq 0$ it follows from (1.14) and (1.15) that

$$c_1 y_1(x, s) + c_2 y_2(x, s) \notin L^2(-\infty, \infty)$$

unless $c_1 = c_2 = 0$. So y is the unique L^2 solution of $l(y) - \lambda y = f$ and $y \in D$ follows easily from direct considerations.

COROLLARY 2.1. *L is a closed operator.*

Proof. If $y_n \in D$, $y_n \to y$, and $Ly_n \to f$ both in L^2, then we must show that $y \in D$ and $Ly = f$. As $W(s) \not\equiv 0$ there is an $s_0 = \sigma_0 + i\tau_0$ with $\tau_0 > 0$ and $W(s_0) \neq 0$, and we have

$$y_n(x) = \int_{-\infty}^{\infty} K(x, \xi, s_0)[Ly_n - s_0^2 y_n]\, d\xi.$$

Thus

$$v(x) = \int_{-\infty}^{\infty} K(x, \xi, s_0)[f - s_0^2 y]\, d\xi$$

almost everywhere.

As the convergence is in L^2 we may replace the limit y by an equivalent member of L^2 so that the above relation is an equality and we have $y \in D$ and $Ly = f$.

LEMMA 2.1. *The adjoint L^* of L is the operator with domain D defined for $y \in D$ by*

$$L^* y = -y'' + \overline{g(x)} y = l^*(y).$$

Proof. We first note that if $y \in D$ then y and y' both tend to 0 as x approaches either $+\infty$ or $-\infty$. This follows from

$$y(x) = \int_{-\infty}^{\infty} K(x, \xi, s_0)[Ly - s_0^2 y]\, d\xi \qquad \text{for} \quad y \in D.$$

Thus if $z \in L^2$ and there exists $z^* \in L^2$ such that $(Ly, z) = (y, z^*)$ for all $y \in D$ then

$$(y, z^*) - (s_0^2 y, z) = (y, z^* - \overline{s_0^2} z)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} K(x, \xi, s_0)(Ly(\xi) - s_0^2 y(\xi))\, d\xi \right] (\overline{z^*(x)} - s_0^2 \overline{z(x)})\, dx$$

$$= \int_{-\infty}^{\infty} (Ly(\xi) - s_0^2 y(\xi)) \overline{\left[\int_{-\infty}^{\infty} \overline{K(x, \xi, s_0)} (z^*(x) - \overline{s_0^2} z(x)) dx \right]} d\xi$$

$$= (Ly - s_0^2 y, z_1).$$

However $(Ly - s_0^2 y, z) = (y, z^*) - (s_0^2 y, z) = (Ly - s_0^2 y, z_1)$ so $z = z_1$ almost everywhere. Thus the domain D^* of L^* consists of functions in L^2 of the form

$$z_1(x) = \int_{-\infty}^{\infty} \overline{K(\xi, x, s_0)} (z^*(\xi) - \overline{s_0^2} z_1(\xi))\, d\xi$$

and $L^* z_1 = z^*$. This implies that $z_1 \in D$, and an easy calculation shows that $-z_1'' + \overline{g(x)} z_1 = z^*$, which completes the proof.

THEOREM 2.2. *The spectrum of L consists of an at most countable, bounded set of characteristic values and a continuous spectrum on the non-negative real axis $\lambda \geqq 0$.*

Proof. If λ is not real and non-negative we have seen that λ is in the resolvent set of L unless $W(\lambda^{\frac{1}{2}}) = 0$. If $W(\lambda^{\frac{1}{2}}) = 0$, λ is obviously a characteristic value with characteristic function $y_1(x, \lambda^{\frac{1}{2}})$. Corollary 1.1 immediately yields the statement about the point spectrum except for $\lambda = 0$.

If $y(x)$ is a characteristic function for $\lambda = 0$ then by using a representation in terms of the Green's function it is easy to see that y is bounded and approaches 0 at $\pm\infty$. Thus

$$y(x) = -\int_x^\infty (x-\xi)g(\xi)y(\xi)\,d\xi$$

and using Lemma 1.1 we see that $y = 0$ and thus $\lambda = 0$ cannot be a characteristic value.

We see that $(L - \sigma^2)^{-1}$ is not bounded for $\sigma > 0$ by attempting to construct $y_0 \in L^2(-\infty, \infty)$ such that

$$l(y_0) - \sigma^2 y_0 = \begin{cases} \overline{y(x, \sigma)} & |x| < a \\ 0 & |x| > a. \end{cases}$$

Thus we see that the positive real axis is in the spectrum, and as the spectrum is closed zero belongs to the spectrum.

To see that the residual spectrum is empty we note that it must lie in the non-negative real axis and if σ^2 is in the residual spectrum it is in the point spectrum of L^*. This would mean that $l^*(y) = \sigma^2 y$ has a solution belonging to L^2. Taking the conjugate of this solution we see that σ^2 lies in the point spectrum of L, which contradicts the assumption.

3. An Expansion of the Green's Function. We shall first use the Cauchy Integral to obtain an expansion of the Green's function, and then use this to obtain our expansion theorem.

Let $C_{R,\delta}$ denote the contour in the s-plane consisting of the straight line $\tau = \delta > 0$ from $\sigma = -(R^2 - \delta^2)^{\frac{1}{2}}$ to $\sigma = (R^2 - \delta^2)^{\frac{1}{2}}$, and the circular arc $s = Re^{i\theta}$ from $\theta = \eta = \sin^{-1}\delta/R$ to $\theta = \pi - \eta$. We choose δ and R_0 so that if λ_0 is a characteristic value $Im\,\lambda_0^{\frac{1}{2}} \neq \delta$ and $R_0^2 > |\lambda_0|$; and consider

$$I_{R,\delta} = \oint_{C_{R,\delta}} \frac{K(x, \xi, s)}{s^2 - \lambda}\, s\, ds \tag{3.1}$$

for $R > R_0$, and $\lambda^{\frac{1}{2}}$ within the contour.

In evaluating (3.1) by residues we see that the singularities of the integrand occur at $s = \lambda^{\frac{1}{2}}$, and at the square roots of the characteristic values. If λ_1, $\lambda_2, \ldots$ are the characteristic values arranged in order so that $Im\,\lambda_1^{\frac{1}{2}} > Im\,\lambda_2^{\frac{1}{2}} > \ldots$; we see that for any δ and $R > R_0$ there is an integer $n(\delta)$ such that $Im\,(\lambda_{n(\delta)})^{\frac{1}{2}} > \delta > Im\,(\lambda_{n(\delta)+1})^{\frac{1}{2}}$, and the value of $I_{R,\delta}$ is thus independent of R for $R > R_0$. As $K(x, \xi, s)$ is the ratio of two functions, each of which is analytic for $\tau > 0$ we see that the singularities at

$$s^2 = \lambda_1, \lambda_2, \ldots$$

must be poles. If we have

$$K(x, \xi, s) = \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(s^2 - \lambda_i)^{-p} + F(x, \xi, s)$$

for s^2 sufficiently close to λ_i where F is analytic in s at $s = \lambda_i^{\frac{1}{2}}$, it is easily seen that the residue of the integrand in (3.1) at $s = \lambda_i^{\frac{1}{2}}$ is

$$-\tfrac{1}{2} \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(\lambda - \lambda_i)^{-p}.$$

Since the residue at $s = \lambda^{\frac{1}{2}}$ is $\frac{1}{2}G(x, \xi, \lambda)$ we have

$$I_{R,\delta} = \pi i G(x, \xi, \lambda) - \pi i \sum_{i=1}^{n(\delta)} \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(\lambda - \lambda_i)^{-p}. \tag{3.2}$$

Now if we evaluate $I_{R,\delta}$ directly we have

$$I_{R,\delta} = \int_{\eta}^{\pi-\eta} \frac{K(x, \xi, Re^{i\theta})iR^2e^{2i\theta}}{R^2e^{2i\theta} - \lambda} d\theta + \int_{-(R^2-\delta^2)^{\frac{1}{2}}}^{(R^2-\delta^2)^{\frac{1}{2}}} \frac{(\sigma + i\delta)K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} d\sigma.$$

Since $|K(x, \xi, s)| < K|W(s)|^{-1}\exp[-\tau|\xi - x|]$, and, for $|s|$ sufficiently large, $|W(s)| > |s|$; we see that

$$\left| \int_{\eta}^{\pi-\eta} \frac{K(x, \xi, Re^{i\theta})iR^2e^{2i\theta}}{R^2e^{2i\theta} - \lambda} d\theta \right| < \frac{K}{R}$$

for R sufficiently large, and

$$\left| \frac{(\sigma + i\delta)K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} \right| < \frac{K}{\sigma^2 + 1}.$$

Thus we have

$$\lim_{R\to\infty} I_{R,\delta} = \int_{-\infty}^{\infty} \frac{(\sigma + i\delta)K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} d\sigma,$$

where the integral converges absolutely and uniformly for $Im\, \lambda^{\frac{1}{2}} > \delta_1 > \delta$ and any x, ξ. Combining this result with (3.2) and the remark that $I_{R,\delta}$ is independent of R for $R > R_0$ we have the following theorem.

THEOREM 3.1. *With the notation introduced above, and under the restrictions on δ introduced above, we have*

$$G(x, \xi, \lambda) = \sum_{i=1}^{n(\delta)} \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(\lambda - \lambda_i)^{-p} + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(\sigma + i\delta)K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} d\sigma. \tag{3.3}$$

It will be convenient to write this in a different form, which will exhibit the symmetry between L and L^*.

LEMMA 3.1. *There are sets of functions $\chi_j^{(i)}(x)$, $\psi_j^{(i)}(x)$ for $j = 0, 1, \ldots, m_i - 1$ such that*

$$l(\chi_j^{(i)}) - \lambda_i\chi_j^{(i)} = \chi_{j-1}^{(i)}, \quad l^*(\psi_j^{(i)}) - \bar{\lambda}_i\psi_j^{(i)} = \psi_{j-1}^{(i)} \tag{3.4}$$

for $j = 0, 1, \ldots, m_i - 1$, where $\chi_{-1}^{(i)} \equiv \psi_{-1}^{(i)} \equiv 0$, and

$$G_p^{(i)}(x, \xi) = -\sum_{j=0}^{m_i-p} \chi_j^{(i)}(x)\, \overline{\psi_{m_i-p-j}^{(i)}(\xi)}. \tag{3.5}$$

Also, each $\chi_j^{(i)}$ *and* $\psi_j^{(i)}$ *is bounded by* $K \exp[-\tau_i|x|/2]$ where $\tau_i = Im\, \lambda_i^{\frac{1}{2}}$, *and*

$$(3.6) \qquad (\chi_j^{(i)}, \psi_r^{(k)}) = \int_{-\infty}^{\infty} \chi_j^{(i)}(x)\, \overline{\psi_r^{(k)}(x)}\, dx = \delta_{ik}\delta_{m_i-j-1\, r}.$$

Proof. Let $\Gamma(x, \xi, \lambda)$ denote the function which is given by $\bar{y}(x, \lambda^{\frac{1}{2}})y(\xi, \lambda^{\frac{1}{2}})$ for $\xi < x$ and by $\bar{y}(\xi, \lambda^{\frac{1}{2}})y(x, \lambda^{\frac{1}{2}})$ for $x < \xi$. This is trivially an entire function of λ and $G(x, \xi, \lambda) - \Gamma(x, \xi, \lambda)$ is of class C^2 as a function of x or ξ. Thus if C_i is a circle with centre at λ_i, enclosing no other points of the spectrum of L we see that

$$G_p^{(i)}(x, \xi) = \frac{1}{2\pi i}\oint_{C_i} (\lambda - \lambda_i)^{p-1}[G(x, \xi, \lambda) - \Gamma(x, \xi, \lambda)]\, d\lambda.$$

From this we see that $G_p^{(i)}(x, \xi)$ is of class C^2 as a function of x or ξ, and from this it is easily seen that as a function of x, $l(G_p^{(i)}) - \lambda_i G_p^{(i)} = G_{p+1}^{(i)}$, and as a function of ξ

$$l^*(\overline{G_p^{(i)}}) - \overline{\lambda_i}\, \overline{G_p^{(i)}} = \overline{G_{p+1}^{(i)}},$$

where

$$G_{m_i+1}^{(i)} \equiv 0.$$

From these equations it follows that $G_p^{(i)}$ can be given in the form (3.5) by functions $\chi_j^{(i)}$ and $\psi_j^{(i)}$ satisfying (3.4).

Now we also have

$$G_p^{(i)}(x, \xi) = \frac{1}{2\pi i}\oint_{C_i} (\lambda - \lambda_i)^{p-1} G(x, \xi, \lambda)\, d\lambda,$$

from which we see, by taking the radius of C_i sufficiently small, that $|G_p^{(i)}(x, \xi)| < K \exp[-\tau_i|x - \xi|/2]$. Using this and an induction we find that $\chi_j^{(i)}$ and $\psi_j^{(i)}$ are both bounded by $K \exp[-\tau_i|x|/2]$. In order to prove (3.6) we note that

$$(L\chi_j^{(i)}, \psi_r^{(k)}) = \lambda_i(\chi_j^{(i)}, \psi_r^{(k)}) + (\chi_{j-1}^{(i)}, \psi_r^{(k)})$$

and

$$(\chi_j^{(i)}, L^*\psi_r^{(k)}) = \lambda_r(\chi_j^{(i)}, \psi_r^{(k)}) + (\chi_j^{(i)}, \psi_{r-1}^{(k)}),$$

so that

$$(\lambda_i - \lambda_r)(\chi_j^{(i)}, \psi_r^{(k)}) = (\chi_j^{(i)}, \psi_{r-1}^{(k)}) - (\chi_{j-1}^{(i)}, \psi_r^{(k)}).$$

From the fact that $\chi_{-1}^{(i)} \equiv \psi_{-1}^{(k)} \equiv 0$ it follows that $(\chi_0^{(i)}, \psi_0^{(k)}) = 0$ for $k \neq i$ so an easy induction yields $(\chi_j^{(i)}, \psi_r^{(k)}) = 0$ if $k \neq i$. To deal with the case $k = i$ note that

$$(l - \lambda)\left(-\sum_{k=0}^{j} \chi_k^{(i)}(x)(\lambda - \lambda_i)^{k-j-1}\right) = \chi_j^{(i)}(x);$$

and thus

$$-\sum_{k=0}^{j} \chi_k^{(i)}(x)(\lambda-\lambda_i)^{k-j-1} = \int_{-\infty}^{\infty} G(x,\xi,\lambda)\chi_j^{(i)}(\xi)\,d\xi,$$

as the right hand side is the unique L^2 solution of $l(y) - \lambda y = \chi_j^{(i)}$ and the left hand side is such a solution. Thus

$$-\sum_{k=0}^{j} \chi_k^{(i)}(x)(\lambda-\lambda_i)^{k-j-1} = \sum_{p=1}^{m_i} (\lambda-\lambda_i)^{-p} \int_{-\infty}^{\infty} G_p^{(i)}(x,\xi)\chi_j^{(i)}(\xi)\,d\xi + F(x,\lambda)$$

where $F(x, \lambda)$ is analytic at $\lambda = \lambda_i$. From this we have

$$\sum_{k=0}^{m_i-p} \chi_k^{(i)}(x)(\chi_j^{(i)}, \psi_{m_i-p-k}^{(i)}) = -\int_{-\infty}^{\infty} G_p^{(i)}(x,\xi)\chi_j^{(i)}(\xi)\,d\xi = \begin{cases} 0 & p > j+1 \\ \chi_{j+1-p}^{(i)}(x) & p < j+1. \end{cases}$$

As

$$\chi_0^{(i)}, \dots, \chi_{m_i-1}^{(i)}$$

are easily seen to be linearly independent we have

$$(\chi_j^{(i)}, \psi_{m_i-p-k}^{(i)}) = \begin{cases} 0 & p > j+1 & \\ 0 & p < j+1 & k \neq j+1-p \\ 1 & p < j+1 & k = j+1-p. \end{cases}$$

Combining this with the result for functions corresponding to different characteristic values we have (3.6).

We might remark that it can be shown that

$$\chi_j^{(i)}(x) = \sum_{k=0}^{j} a_{j-k}^{(i)} \frac{1}{k!} \frac{d^k}{d\lambda^k} y_1(x, \lambda^{\frac{1}{2}})_{\lambda=\lambda_i}$$

for suitable constants $a_k^{(i)}$, and a similar result for $\psi_j^{(i)}$.

4. The Expansion in Case I. In order to obtain an expansion analogous in form to that which holds when L is self-adjoint, we must modify the integral in (3.3) so that it involves only solutions of $l(y) = \lambda y$ for $\lambda > 0$. The obvious way to do this is to evaluate the limit as $\delta \to 0$, but this may lead to two difficulties:

(i) $n(\delta)$ may become infinite and the discrete portion of the expansion may diverge.

(ii) The integral in (3.3) may not exist for $\delta = 0$ if W has real zeros.

Although the convergence difficulties do not arise, $n(\delta)$ may become infinite even if L is self-adjoint. We shall construct two examples, both with $g(x) = 0$ for $|x| > b$, to show that W can have real zeros of sufficiently high order that the integral in (3.3) will not exist even as a principal value for $\delta = 0$.

As $g(x) = 0$ for $|x| > b$, $W(s) = -e^{isb}[y_1'(-b, s) + isy_1(-b, s)]$ and if W is to have a zero of order m at $s = s_0$ we find that we must have

$$y_1^{(k)\prime}(-b, s_0) + is_0 y_1^{(k)}(-b, s_0) + iy_1^{(k-1)}(-b, s_0) = 0, \qquad k = 0, 1, \ldots, m-1$$

where

$$y_1(x, s) = \sum_{n=0}^{\infty} y_1^{(n)}(x, s_0)(s - s_0)^n \quad \text{and} \quad y_1^{(-1)}(x, s_0) = 0.$$

Example I. Third order zero at $s = 0$. We set $y_1(x, 0) = e^{i\theta(x)}$ so that $g(x) = i\theta''(x) - [\theta'(x)]^2$ for $|x| < b$ and require that $\theta \in C^\infty$, $\theta(-b) = -\frac{1}{2}\pi$, $\theta(b) = 2\pi$, $\theta^{(n)}(-b) = \theta^{(n)}(b) = 0$ for $n > 0$, and $\int_{-b}^{b} \sin 2\theta(x)dx = 0$.

Example II. Second order zero at $s = 1$. We set $y_1(x, 1) = e^{ib}f(x)$ so that $g(x) = 1 + f''(x)/f(x)$ for $|x| < b$ and require that $f(x) \neq 0$ for $|x| < b$, $f^{(n)}(b) = i^n$, $f^{(n)}(-b) = (-i)^n f(-b)$, $f \in C^\infty$, and

$$[f(-b)]^2 = -1 + 2i\int_{-b}^{b}[f(x)]^2dx.$$

Here we may obtain an explicit $f(x)$ as a polynomial if we do not require that $g \in C^\infty$ at $x = \pm b$, that is, set

$$4b^3 f(x) = b(2 - ib)\,[\alpha(x-b)^2 + (x+b)^2] + x(1 - ib)\,[\alpha(x-b)^2 - (x+b)^2]$$

and choose α so that

$$\alpha^2 = -1 + 2i\int_{-b}^{b}[f(x)]^2dx.$$

Thus, even if g is a C^∞ function of compact support, the integral in (3.3) may still not exist for $\delta = 0$. We shall now add the assumptions of Case I that, for sufficiently small $|s|$, $|W(s)| > K|s|$ and that W has no real zeros except possibly $s = 0$.

With these assumptions $n(\delta)$ must remain finite as $\delta \to 0$ and we shall suppose that $n(\delta) = n$ (its maximum) for $\delta < \delta_0$. Thus for $\delta < \delta_0$ the integral in (3.3) is independent of δ, and for $\delta < \frac{1}{2}\delta_0$ the integrand is bounded by $K(\sigma^2 + 1)^{-1}$ where K is independent of δ. Thus we may set $\delta = 0$ in (3.3) to obtain

$$G(x, \xi, \lambda) = \sum_{i=1}^{n} \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(\lambda - \lambda_i)^{-p} + \frac{1}{\pi i}\int_0^\infty \frac{[\sigma K(x, \xi, \sigma) - \sigma K(x, \xi, -\sigma)]}{\sigma^2 - \lambda}\,d\sigma. \tag{4.1}$$

LEMMA 4.1. *We have*

$$\sigma K(x, \xi, \sigma) - \sigma K(x, \xi, -\sigma) = \frac{2i\sigma^2}{W(\sigma)W(-\sigma)}\,[y_1(x, \sigma)y_1(\xi, -\sigma) + y_2(x, \sigma)y_2(\xi, -\sigma)].$$

Proof. From the definition of $K(x, \xi, \sigma)$ we have

$$\sigma K(x, \xi, \sigma) - \sigma K(x, \xi, -\sigma) = \frac{\sigma}{W(\sigma)W(-\sigma)}\begin{cases} W(-\sigma)y_1(x, \sigma)y_2(\xi, \sigma) - W(\sigma)y_1(x, -\sigma)y_2(\xi, -\sigma), & \xi < x \\ W(-\sigma)y_1(\xi, \sigma)y_2(x, \sigma) - W(\sigma)y_1(\xi, -\sigma)y_2(x, -\sigma), & x < \xi. \end{cases}$$

If we denote $y_i(x, \pm\sigma)y_j'(x, \pm\sigma) - y_i'(x, \pm\sigma)y_j(x, \pm\sigma)$ by $W(y_i(\pm\sigma), y_j(\pm\sigma))$ and note that

$$W(y_1(\sigma), y_1(-\sigma)) = -W(y_2(\sigma), y_2(-\sigma)) = -2i\sigma,$$

then

$$y_2(\xi, \sigma) = \frac{2i\sigma}{W(-\sigma)} y_1(\xi, -\sigma) - \frac{W(y_2(+\sigma), y_1(-\sigma))}{W(-\sigma)} y_2(\xi, -\sigma)$$

and

$$y_1(x, -\sigma) = \frac{W(y_1(-\sigma), y_2(+\sigma))}{W(\sigma)} y_1(x, \sigma) - \frac{2i\sigma}{W(\sigma)} y_2(x, +\sigma).$$

Thus

$$\begin{aligned} W(-\sigma)&y_1(x, \sigma)y_2(\xi, \sigma) - W(\sigma)y_1(x, -\sigma)y_2(\xi, -\sigma) \\ &= 2i\sigma y_1(x, \sigma)y_1(\xi, -\sigma) - W(y_2(+\sigma), y_1(-\sigma))y_1(x, \sigma)y_2(\xi, -\sigma) \\ &- W(y_1(-\sigma), y_2(+\sigma))y_1(x, \sigma)y_2(\xi, -\sigma) + 2i\sigma y_2(x, \sigma)y_2(\xi, -\sigma) \\ &= 2i\sigma[y_1(x, \sigma)y_1(\xi, -\sigma) + y_2(x, \sigma)y_2(\xi, -\sigma)]. \end{aligned}$$

A similar computation yields the same result for $x < \xi$.

COROLLARY 4.1. *$G(x, \xi, \lambda)$ can be written in the form*

$$G(x, \xi, \lambda) = -\sum_{i=1}^{n} \sum_{p=1}^{m_i} \sum_{j=0}^{m_i-p} \chi_j^{(i)}(x)\, \overline{\psi_{m_i-p-j}^{(i)}(\xi)}\,(\lambda - \lambda_i)^{-p} + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} \frac{\phi_i(x, \sigma)\overline{\theta_i(\xi, \sigma)}}{\sigma^2 - \lambda}\, d\sigma \tag{4.2}$$

where

$$\phi_i(x, \sigma) = \frac{\sigma}{W(\sigma)} y_i(x, \sigma) \quad \text{and} \quad \theta_i(x, \sigma) = \frac{\sigma}{W^*(\sigma)} y_i^*(x, \sigma).$$

Here * denotes the corresponding quantity associated with the adjoint equation.

Proof. As

$$y_1^*(x, \sigma) = e^{i\sigma x} - \int_x^\infty \frac{\sin\sigma(x - \xi)}{\sigma} \overline{g(\xi)} y_1^*(\xi, \sigma)\, d\xi$$

it follows immediately that

$$y_1^*(x, \sigma) = \overline{y_1(x, -\sigma)}.$$

Similarly

$$y_2^*(x, \sigma) = \overline{y_2(x, -\sigma)}$$

so

$$W^*(\sigma) = \overline{W(-\sigma)}$$

and using these relations with Lemma 4.1 in (4.1) we obtain (4.2).

We are now in a position to prove an expansion theorem.

THEOREM 4.1. *If $f \in L^p$ for $p > 1$ then*

$$f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \chi_j^{(i)}(x)(f, \psi_{m_i-j-1}^{(i)}) + (l-\lambda)\frac{2}{\pi}\int_{-\infty}^{\infty} f(\xi) \int_0^{\infty} \sum_{i=1}^{2} \frac{\phi_i(x,\sigma)\overline{\theta_i(\xi,\sigma)}}{\sigma^2-\lambda}\, d\sigma\, d\xi, \tag{4.3}$$

almost everywhere, for any λ not in the spectrum.

Proof. If $f \in L^p$ for $p > 1$ it is easily seen that if λ is not in the spectrum of L then

$$\int_{-\infty}^{\infty} G(x, \xi, \lambda) f(\xi)\, d\xi$$

exists and is the unique L^p solution of $l(y) - \lambda y = f$. Using (4.2) to calculate the integral and applying $l - \lambda$ we immediately obtain (4.3).

The last term of (4.3) is not in a very convenient form, but in order to simplify it we must impose some restrictions on f. If $f \in L^1$ then the order of integration in the last term may be inverted, and setting

$$f_i(\sigma) = \int_{-\infty}^{\infty} f(x)\, \overline{\theta_i(x,\sigma)}\, dx$$

we obtain

$$f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \chi_j^{(i)}(x)(f, \psi_{m_i-j-1}^{(i)}) + (l-\lambda)\frac{2}{\pi}\int_0^{\infty} \sum_{i=1}^{2} \frac{\phi_i(x,\sigma) f_i(\sigma) d\sigma}{\sigma^2-\lambda}. \tag{4.4}$$

We define D_1 to be the class of functions $f \in L^1$ with derivatives which are absolutely continuous on every finite interval and such that $l(f) \in L^1$. Then for $f \in D_1$ choose $-a^2 < 0$ not in the spectrum of L and set $h = l(f) + a^2 f$. Then it is easily seen that f and f' approach 0 as x approaches $\pm\infty$ so

$$\int_{-\infty}^{\infty} h(x)\, \overline{\theta_i(x,\sigma)}\, dx = (\sigma^2 + a^2)\int_{-\infty}^{\infty} f(x)\, \overline{\theta_i(x,\sigma)}\, dx$$

and thus

$$|f_i(\sigma)| < K(\sigma^2 + a^2)^{-1}\int_{-\infty}^{\infty} |h(x)| dx.$$

So for $f \in D_1$ the operation of $l - \lambda$ in (4.4) may be perfomed under the integral sign to obtain

$$f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \chi_j^{(i)}(x)(f, \psi_{m_i-j-1}^{(i)}) + \frac{2}{\pi}\int_0^{\infty} \sum_{i=1}^{2} \phi_i(x,\sigma) f_i(\sigma)\, d\sigma. \tag{4.5}$$

We also have an analogue of the Parseval equality, and a corresponding expansion theorem associated with L^*.

THEOREM 4.2. *As well as* (4.5) *we have, for* $f \in D_1$

$$(4.6)\quad f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \psi_j^{(i)}(x)(f, \chi_{m_i-j-1}^{(i)}) + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} \theta_i(x,\sigma) f_i^*(\sigma)\, d\sigma$$

where

$$f_i^*(\sigma) = \int_{-\infty}^{\infty} f(x)\overline{\phi_i(x,\sigma)}\, dx;$$

and if $f, g \in D_1$

$$(4.7)\quad (f,g) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} (f, \psi_{m_i-j-1}^{(i)})(\chi_j^{(i)}, g) + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} f_i(\sigma)\overline{g_i^*(\sigma)}\, d\sigma$$

$$= \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} (f, \chi_{m_i-j-1}^{(i)})(\psi_j^{(i)}, g) + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} f_i^*(\sigma)\overline{g_i(\sigma)}\, d\sigma.$$

Proof. The proof of (4.6) is analogous to that of (4.5) and to obtain the two forms of (4.7) we note that if $f \in D_1$ it is in L^2 as well as L^1 and take the inner products of (4.5) and (4.6) with g. In doing this the order of integration in the last term can trivially be inverted to obtain the results.

5. The Expansion in Case II. Here we may assume that for some $a > 0$, $e^{a|x|}g(x) \in L^1$, but it is a consequence of this about the zeros of $W(s)$ which we use. If $e^{a|x|}g(x) \in L^1$; $y_1(x,s)$, $y_2(x,s)$, and thus $W(s)$ are analytic for $\tau > -\tfrac{1}{2}a$. In conjunction with Corollary 1.1 this implies that W has only a finite number of zeros in $\tau \geqslant 0$, and this is the assumption we make.

Suppose that the real zeros of W are $\sigma_1, \sigma_2, \ldots, \sigma_q$, and perhaps $\sigma_0 = 0$, arranged so that $0 = \sigma_0^2 < \sigma_1^2 < \ldots < \sigma_q^2$. Choose r so that $r < 2(\sigma_{i+1}^2 - \sigma_i^2)$ for $i = 0, 1, \ldots, q-1$ and so that $r < \min\,[Im(\lambda_i^{\frac{1}{2}})]^2$ for all λ_i in the point spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. We define the contour C by $\tau = f(\sigma)$ where $f(\sigma) = 0$ for $|\sigma^2 - \sigma_j^2| \geqslant r$ (j running from 0 to q or 1 to q according as $W(0)$ is or is not 0), and $f(\sigma) = (r^2 - (\sigma^2 - \sigma_j^2)^2)^{\frac{1}{2}}$ for $|\sigma^2 - \sigma_j^2| \leqslant r$. Now if $\delta < \min[Im(\lambda_i^{\frac{1}{2}})]$ the integral in (3.3) along $\tau = \delta$ is equal to the integral along C.

As the portion of C lying along $\tau = 0$ is symmetric about 0 we may transform it to an integral over $L = \{\sigma | \sigma \geqslant 0,\ |\sigma^2 - \sigma_j^2| \geqslant r\}$ with the same integrand as (4.1). The sum of the integrals over the indentations about σ_i and $-\sigma_i$ can be transformed by a change of variable into $\frac{1}{2}\oint_{C_i} G(x,\xi,\mu)(u-\lambda)^{-1} d\mu$ where C_i is the circle of radius r about σ_i^2. Note that $G(x,\xi,\mu)$ is discontinuous where C_i crosses the real axis. This proves

THEOREM 5.1. *Under the hypothese of Case II we have*:

$$(5.1)\quad G(x,\xi,\lambda) = -\sum_{i=1}^{n} \sum_{p=1}^{m_i} \sum_{j=0}^{m_i-p} \chi_j^{(i)}(x)\overline{\psi_{m_i-p-j}^{(i)}(\xi)}\,(\lambda-\lambda_i)_{-p}$$

$$+ \frac{2}{\pi}\int_L \sum_{i=1}^{2} \frac{\phi_i(x,\sigma)\overline{\theta_i(\xi,\sigma)}}{\sigma^2-\lambda}\, d\sigma + \sum_j \frac{1}{2\pi i}\oint_{C_j} \frac{G(x,\xi,\mu)}{\mu-\lambda}\, d\mu.$$

The only change from (4.2) is that the integral in the second term is taken over L rather than over $[0, \infty)$, and an extra sum is introduced. The other

results carry over in the same way, replacing $[0, \infty)$ by L and adding a new summation. We shall merely indicate the forms these sums must take by considering a sample term

$$\frac{1}{2\pi i}\oint_{C_j} \frac{G(x, \xi, \mu)}{\mu - \lambda}\, d\mu.$$

In expanding $f \in D_1$ we have

$$\begin{aligned}(L - \lambda) \int_{-\infty}^{\infty} \frac{f(\xi)}{2\pi i} \oint_{C_j} \frac{G(x, \xi, \mu)}{\mu - \lambda}\, d\mu\, d\xi \\ &= (L - \lambda) \frac{1}{2\pi i} \oint_{C_j} \frac{1}{\mu - \lambda} \int_{-\infty}^{\infty} G(x, \xi, \mu) f(\xi)\, d\xi\, d\mu \\ &= \frac{1}{2\pi i} \oint_{C_j} \int_{-\infty}^{\infty} G(x, \xi, \mu) f(\xi)\, d\xi\, d\mu,\end{aligned}$$

and in the analogue of the Parseval equality we have

$$\begin{aligned}\frac{1}{2\pi i} \int_{-\infty}^{\infty} \overline{g(x)} \oint_{C_j} \int_{-\infty}^{\infty} G(x, \xi, \mu) f(\xi)\, d\xi\, d\mu\, dx \\ &= \frac{1}{2\pi i} \oint_{C_j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, \xi, \mu) f(\xi) \overline{g(x)}\, d\xi\, dx\, d\mu.\end{aligned}$$

The formulas arising from L^* are the same with G replaced by G^*.

A transformation of

$$H_j(x) = \frac{1}{2\pi i} \oint_{C_j} \int_{-\infty}^{\infty} G(x, \xi, \mu) f(\xi)\, d\xi\, d\mu$$

yields

$$H_j(x) = [L - \sigma_j^2]^{-t} \frac{2}{\pi} \int_{(\sigma_j^2 - r)^{\frac{1}{2}}}^{(\sigma_j^2 + r)^{\frac{1}{2}}} (\sigma^2 - \sigma_j^2)^t \sum_{k=1}^{2} \phi_k(x, \sigma) f_k(\sigma)\, d\sigma$$

where t is sufficiently large that $(s^2 - \sigma_j^2)^t K(x, \xi, s)$ is continuous at $\pm \sigma_j$, but one cannot carry the (unbounded) operator $[L - \sigma_j^2]^{-t}$ under the integral sign. In particular cases one can also evaluate the limit as $r \to 0$ in terms of the principal value of $\int_0^{\infty} \ldots d\sigma$ and a sum of terms which appear to involve characteristic functions, but do not.

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ON THE PRODUCT OF TWO KUMMER SERIES

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1. Introduction. Let $\alpha, \beta, \mu, \nu, z$ be complex numbers such that 2μ and 2ν are not negative integers. Using the notation of (**4**) for generalized hypergeometric series, we set

$$(1) \qquad \phi(z) = {}_1F_1\left[\begin{matrix}\mu + \frac{1}{2} - \alpha; -z \\ 2\mu + 1\end{matrix}\right] {}_1F_1\left[\begin{matrix}\nu + \frac{1}{2} - \beta; z \\ 2\nu + 1\end{matrix}\right]$$

and define $a_n = a_n(\alpha, \beta, \mu, \nu)$ by

$$(2) \qquad \phi(z) = \sum_{n=0}^{\infty} a_n z^n.$$

It is evident that the function $\phi(z)$ does not change if the parameters are subjected to the transformation

$$S: (\alpha, \beta, \mu, \nu; z) \to (\beta, \alpha, \nu, \mu; -z);$$

this merely interchanges the two factors in (1). The function $\phi(z)$ also admits of a further, and less evident, transformation. Applying Kummer's transformation (**4**, 6.3 (7)) to the two series on the right of (1), we find that $\phi(z)$ can also be written as follows:

$$(3) \qquad \phi(z) = {}_1F_1\left[\begin{matrix}\mu + \frac{1}{2} + \alpha; z \\ 2\mu + 1\end{matrix}\right] {}_1F_1\left[\begin{matrix}\nu + \frac{1}{2} + \beta; -z \\ 2\nu + 1\end{matrix}\right].$$

Thus, $\phi(z)$ is invariant under the transformation

$$T: (\alpha, \beta, \mu, \nu; z) \to (-\alpha, -\beta, \mu, \nu; -z).$$

It follows from the above that the coefficients a_n satisfy

$$(4) \qquad a_n(\beta, \alpha, \nu, \mu) = a_n(-\alpha, -\beta, \mu, \nu) = (-1)^n a_n(\alpha, \beta, \mu, \nu).$$

These relations of symmetry are not completely mirrored in the representation (1) of the generating function of the a_n. While it is true that the function $\phi(z)$ as a whole is invariant under both transformations S and T, the particular factorization (1) is invariant only under S but not under T.

The primary objective of this paper is the derivation of a generating function for the coefficients a_n which renders explicit both relations (4). Widening the scope of our problem somewhat, we shall in fact derive a *complete set of generating functions* of the form

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$$\psi(z) = \sum_{n=0}^{\infty} c_n a_n z^n, \tag{5}$$

where the c_n are quotients of products of factorials and the functions $\psi(z)$ are products of two (generalized) hypergeometric series. Any generating function of this type belongs to exactly one of four classes according to the invariance of the factorization of $\psi(z)$ under none, exactly one, or both of the transformations S and T. The set of generating functions to be given below is complete in the sense that each class is represented in it. Applying the transformations under which the factorizations are not invariant, we shall obtain $4 + 2 + 2 + 1$ different factorizations for generating functions of the form (5).

Our results do not answer completely the following question raised by a referee. Do there exist generating functions of the form (5) with factorizations which are invariant under ST but not under both S and T? It is easy to show that any factorization left invariant under ST and one of the transformations S and T is left invariant also under the other, but our method fails to show whether there exists a generating function with a factorization which, although invariant under ST, is changed by both S and T.

2. Representations of a_n in terms of terminating ${}_3F_2$. By Cauchy multiplication of the two series on the right of (2) we get the expression

$$a_n = \frac{(\nu + \frac{1}{2} - \beta)_n}{(2\nu + 1)_n\, n!}\, {}_3F_2\left[\begin{matrix} -2\nu - n, \mu + \frac{1}{2} - \alpha, -n; \\ 2\mu + 1, -\nu + \beta - n + \frac{1}{2} \end{matrix}\right] \tag{6}$$

We shall now utilize some results of a theory due to Whipple on transformations of functions ${}_3F_2$ with unit argument (2, chapter III). According to Whipple, any terminating ${}_3F_2$ can be represented as a product of factorials and a terminating ${}_3F_2$ in eighteen different ways. We divide the resulting eighteen representations of a_n into four classes, according to whether they are invariant under none, exactly one, or both of the transformations S and T. (Two representations which are obtained from each other by reversing the order of summation in the hypergeometric sum are hereby considered identical.) The representation (6) is typical for the class invariant under S but not T. The following are typical representatives of the other classes:

$$a_n = \frac{(\mu + \nu - \alpha - \beta + 1)_n}{(2\nu + 1)_n\, n!}\, {}_3F_2\left[\begin{matrix} 2\mu + 2\nu + n + 1, \mu + \frac{1}{2} - \alpha, -n; \\ 2\mu + 1, \mu + \nu - \alpha - \beta + 1 \end{matrix}\right]. \tag{7}$$

(not invariant);

$$a_n = \frac{(\mu + \nu - \alpha - \beta + 1)_n (\mu + \frac{1}{2} + \alpha)_n}{(2\mu + 1)_n\, (2\nu + 1)_n\, n!}\, {}_3F_2\left[\begin{matrix} -\mu - \nu - \alpha - \beta - n, \mu + \frac{1}{2} - \alpha, -n; \\ \mu + \nu - \alpha - \beta + 1, -\mu - \alpha - n + \frac{1}{2} \end{matrix}\right] \tag{8}$$

(invariant under T);

$$(9)\quad a_n = \frac{(\mu+\frac{1}{2}+\alpha)_n(\nu+\frac{1}{2}-\beta)_n}{(2\mu+1)_n(2\nu+1)_n\, n!}\; {}_3F_2\left[\begin{matrix}\mu+\frac{1}{2}-\alpha, \nu+\frac{1}{2}+\beta, -n;\\ -\mu-\alpha-n+\frac{1}{2}, -\nu+\beta-n+\frac{1}{2}\end{matrix}\right]$$

(invariant under S and T).

Applying to these formulae the transformations under which they are not invariant, we get three new representations of the form (7) and one new representation of each of the forms (6) and (8). This, together with the reversed series, makes up Whipple's total of eighteen series.

3. The complete set of generating functions. We now assert that the following identities hold:

$$(10)\quad {}_1F_0[2\mu+2\nu+1; z]\,{}_3F_2\left[\begin{matrix}\mu+\nu+\frac{1}{2}, \mu+\nu+1, \mu+\frac{1}{2}-\alpha; -\dfrac{4z}{(1-z)^2}\\ 2\mu+1, \mu+\nu-\alpha-\beta+1\end{matrix}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(2\nu+1)_n(2\mu+2\nu+1)_n}{(\mu+\nu-\alpha-\beta+1)_n} a_n z^n$$

(not invariant);

$$(11)\quad {}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}-\alpha; -z\\ 2\mu+1\end{matrix}\right] {}_1F_1\left[\begin{matrix}\nu+\frac{1}{2}-\beta; z\\ 2\nu+1\end{matrix}\right] = \sum_{n=0}^{\infty} a_n z^n$$

(invariant under S);

$$(12)\quad {}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}-\alpha; -z\\ \mu+\nu-\alpha-\beta+1\end{matrix}\right] {}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}+\alpha; z\\ \mu+\nu+\alpha+\beta+1\end{matrix}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(2\mu+1)_n(2\nu+1)_n}{(\mu+\nu+\alpha+\beta+1)_n(\mu+\nu-\alpha-\beta+1)_n} a_n z^n$$

(invariant under T);

$$(13)\quad {}_2F_0[\mu+\tfrac{1}{2}-\alpha, \nu+\tfrac{1}{2}+\beta; -z]\,{}_2F_0[\mu+\tfrac{1}{2}+\alpha, \nu+\tfrac{1}{2}-\beta; z]$$
$$= \sum_{n=0}^{\infty} (2\mu+1)_n(2\nu+1)_n a_n z^n$$

(invariant under both S and T).

Here we have listed for completeness as (11) once again the generating function (2). Applying the transformations under which the factorizations are not invariant, we obtain three new generating functions of the form (10) and a new factorization for each of the functions (11) and (12).

The proof of (12) and (13) follows from (8) and (9) by the equations 4.3 (13) and 4.3 (15) of (4). In order to prove (10), we denote the product on the left of (10) by $\psi(z)$ and observe that

$$\psi(z) = \sum_{p=0}^{\infty} c_p(-4z)^p (1-z)^{-2\mu-2\nu-1-2p},$$

where

$$c_p = \frac{(\mu+\nu+\frac{1}{2})_p(\mu+\nu+1)_p(\mu+\frac{1}{2}-\alpha)_p}{(2\mu+1)_p(\mu+\nu-\alpha-\beta+1)_p\, p!}.$$

Using the binomial expansion and rearranging, we get

$$\begin{aligned}\psi(z) &= \sum_{p=0}^{\infty} c_p(-4z)^p \sum_{q=0}^{\infty} \frac{(2\mu+2\nu+2p+1)_q}{q!} z^q \\ &= \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} (-4)^p c_p \frac{(2\mu+2\nu+2p+1)_{n-p}}{(n-p)!} \\ &= \sum_{n=0}^{\infty} \frac{(2\mu+2\nu+1)_n}{n!} z^n \sum_{p=0}^{n} \frac{(-n)_p(\mu+\frac{1}{2}-\alpha)_p(2\mu+2\nu+n+1)_p}{(2\mu+1)_p(\mu+\nu-\alpha-\beta+1)_p\, p!}.\end{aligned}$$

The inner sum is readily expressed in terms of a_n by (7), and (10) follows.

It will be noted that the generating function (13), which possesses the highest degree of symmetry, diverges for every $z \neq 0$, unless both series on the left terminate. As a formal Cauchy product it retains a meaning in the case of divergence.

4. Identities of Cayley-Orr type. Evidently our results can be interpreted as identities between the coefficients in the expansion of certain products of hypergeometric series. Such identities were first studied by Cayley and Orr (see **2**, chapter X); more recently, the subject has been taken up again by Burchnall and Chaundy **(3)** and the author **(6)**. In fact, the implication (2) → (12) is a confluent form of equation (24) of **(3)**.

5. An application to the product of two Whittaker functions. In this section we shall use the notation of **(4)** for Whittaker functions and Jacobi polynomials. In **(5)** we have proved a result which can be stated thus: Let $\alpha, \beta, \mu, \nu, \rho, \tau$ be arbitrary complex numbers such that none of the numbers $2\mu, 2\nu, 2\mu+2\nu$ is a negative integer, and let a_n be defined by (2). Then the following identity holds:

$$\left(\rho\frac{1-\tau}{2}\right)^{-\mu-\frac{1}{2}} M_{\alpha,\mu}\left(\rho\frac{1-\tau}{2}\right)\cdot\left(\rho\frac{1+\tau}{2}\right)^{-\nu-\frac{1}{2}} M_{\beta,\nu}\left(\rho\frac{1+\tau}{2}\right) = \sum_{n=0}^{\infty} \frac{n!}{(2\mu+2\nu+1+n)_n} a_n P_n^{(2\mu,2\nu)}(\tau)\rho^{-\mu-\nu-1} M_{\alpha+\beta,\mu+\nu+\frac{1}{2}+n}(\rho). \tag{14}$$

We now see that the coefficients a_n can be defined by any of the generating functions given in §3, in particular by the symmetric function (13). Also, making use of the results of Bailey **(1)** on cases where products of two hypergeometric functions can be expressed in terms of a single such function, we now could give a systematic account of those special cases of (14) where a_n can be expressed in terms of factorials only. Most of these cases were noted in **(5)**, using *ad hoc* methods. A further result can be obtained by applying equation (2.10) of **(1)**. We have, provided that 2μ is not an integer,

$$a_n(\alpha,\alpha,\mu,-\mu) = \frac{\mu(\alpha-n/2+\frac{1}{2})_n}{(\mu-n/2)_{n+1}\, n!}. \tag{15}$$

After some simplification we thus obtain from (14)

$$(16)\quad \left(\frac{1-\tau^2}{4}\right)^{-\frac{1}{2}} M_{\alpha,\mu}\left(\rho\frac{1-\tau}{2}\right) M_{\alpha,-\mu}\left(\rho\frac{1+\tau}{2}\right)$$

$$= \Gamma(2\mu+1) \sum_{n=0}^{\infty} \frac{\mu(\alpha - n/2 + \frac{1}{2})_n (2\mu+1)_n}{(2n)!(\mu - n/2)_{n+1}} P_n^{-2\mu}(\tau)\, M_{2\alpha, n+\frac{1}{2}}(\rho).$$

This expansion is a counterpart of the following result, which (in a different notation) can be found in **(5)**:

$$(17)\quad \left(\frac{1-\tau^2}{4}\right)^{-\frac{1}{2}} M_{\alpha,\mu}\left(\rho\frac{1-\tau}{2}\right) M_{\alpha,\mu}\left(\rho\frac{1+\tau}{2}\right)$$

$$= \Gamma(2\mu+1) \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2}-\alpha)_n(\mu+\frac{1}{2}+\alpha)_n}{n!(2\mu+1)_n(4\mu+1+2n)_{2n}} P_{2n+2\mu}^{-2\mu}(\tau) M_{2\alpha, 2\mu+\frac{1}{2}+2n}(\rho).$$

In both (16) and (17) P denotes the Legendre function of the first kind on the cut **(4**, 3.4(6)**)**. The limits of (16) and (17) as $\mu \to 0$ can be written in the form

$$(18)\quad e^{-\rho/2} L_{\alpha-\frac{1}{2}}\left(\rho\frac{1-\tau}{2}\right) L_{\alpha-\frac{1}{2}}\left(\rho\frac{1+\tau}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2(4n)!}(-1)^n(\alpha - n + \tfrac{1}{2})_{2n} P_{2n}(\tau) M_{2\alpha, 2n+\frac{1}{2}}(\rho)$$

where L denotes the Laguerre function and P the Legendre polynomial. For $\alpha = \frac{1}{2}$ all terms on the right of (18) vanish except the first. The relation then becomes trivial, since $L_0 = 1$, $M_{1,\frac{1}{2}}(\rho) = e^{-\rho/2}$.

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ON THE *PROBLÈME DES MÉNAGES*

MAX WYMAN AND LEO MOSER

Introduction. The classical *problème des ménages* asks for the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no husband is next to his own wife.

An outline of the history of the problem to 1946 was given by Kaplansky and Riordan **(11)**. They also presented a bibliography, which is augmented and brought up to date in the bibliography of the present paper.

The first explicit solution of the problem is due to Touchard **(23)** and the simplest derivation of Touchard's formula is due to Kaplansky **(9)**. In the present paper a new explicit solution to the problem is obtained, via an exponential generating function for certain numbers closely related to the ménage numbers and introduced by Cayley **(4)**. Although the new explicit expression is quite complicated, it does lead to some new and deep results concerning the ménage numbers. In particular, it is shown that the usual asymptotic formula for these numbers can actually be used to compute the numbers exactly.

Several other new explicit expressions for the ménage numbers are obtained and one of these suggests a strong conjecture concerning Latin rectangles for which some evidence is presented.

The most extensive published tables of the ménage numbers are those given by Lucas **(13)**. These go up to $n = 25$. In the present paper we present tables which give the numbers up to $n = 65$. These were computed by F. L. Miksa, using a recursion formula of Cayley **(4)**, and checked by means of congruences due to Riordan **(20)**.

1. A Generating Function. Rather than deal directly with the ménage numbers M_n many authors introduce the number U_n defined by

$$M_n = 2\,(n!)\,U_n. \tag{1.1}$$

Further, Cayley **(4)** introduced an auxiliary sequence q_n defined by

$$U_n = q_n - q_{n-2}, \tag{1.2}$$

and showed that the q_n satisfy the recurrence relation

$$q_n = n\,q_{n-1} + q_{n-2} + (-1)^{n-1}(n-2). \tag{1.3}$$

If we introduce the generating function $F(t)$ by

$$F(t) = \sum_{n=0}^{\infty} q_n \frac{t^n}{n!}, \tag{1.4}$$

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then it is easily shown that $F(t)$ is the solution of

$$(1.5)\qquad (1-t)\ddot{F} - 2\dot{F} - F = t\,e^{-t},$$
$$F(0) = \dot{F}(0) = 0,$$

where the "dot" means differentiation with respect to t.

The substitution

$$(1.6)\qquad F = (1-t)^{\frac{1}{2}}y,\ x = 2(1-t)^{\frac{1}{2}}$$

makes (1.5) take the form

$$(1.7)\qquad y'' + x^{-1}y' - (1 + x^{-2})y = \tfrac{1}{2}x(1 - \tfrac{1}{4}x^2)\,e^{(x^2/4-1)},$$
$$y(2) = y'(2) = 0,$$

where the prime denotes differentiation with respect to x. The homogeneous equation is well known and the complementary function can be expressed in terms of the modified Bessel functions as

$$(1.8)\qquad A\,I_1(x) + B\,K_1(x),$$

where A, B are constants.

In order to determine a particular integral $P(x)$ of (1.7), we assume a series solution of the form

$$(1.9)\qquad P(x) = \sum_{n=0}^{\infty} a_n x^{n+3}.$$

Substituting into (1.7) we immediately are led to

$$(1.10)\qquad a_0 = e^{-1}/16,\ a_{2n+1} = 0,$$
$$4a_{2n}(n+1)(n+2) - a_{2n-2} = e^{-1}(1-n)/2^{2n+1}\,n!$$

This recurrence relation is easily solved and our particular solution can be put into the form

$$(1.11)\qquad P(x) = e^{-1}\Big[I_1(x) - \tfrac{1}{2}x\,e^{x^2/4} + 2\sum_{n=1}^{\infty} b_n\,(\tfrac{1}{2}x)^{2n+1}\Big],$$

where

$$b_n = \Big(\sum_{s=1}^{n} s!\Big)/n!(n+1)!\,.$$

Replacing $s!$ by

$$\int_0^{\infty} e^{-z} z^s dz,$$

we find

$$(1.12)\qquad P(x) = e^{-1}\Big[I_1(x) - \tfrac{1}{2}x\,e^{x^2/4} + 2\int_0^{\infty} F(x,z)dz\Big],$$

where $F(x,z) = z\,e^{-z}\,(I_1(x) - z^{\frac{1}{2}}\,I_1\,(x\,z^{-\frac{1}{2}}))/(1-z)$.

If we introduce the principal value of the integral at $z = 1$ we can rearrange the terms so that

$$P(x) = e^{-1}\left[L\, I_1(x) - \tfrac{1}{2}x\, e^{x^2/4} + 2\int_0^\infty G(x, z)dz\right], \tag{1.13}$$

where

$$L = 2\int_0^\infty \frac{e^{-z}}{1-z}dz - 1, \quad G(x, z) = \frac{z^{\frac{1}{2}}e^{-z}I_1(xz^{\frac{1}{2}})}{z-1}. \tag{1.14}$$

Thus the general solution of (1.7) must be of the form

$$y = A\, I_1(x) + B\, K_1(x) + P(x), \tag{1.15}$$

where the constants A, B must be chosen to satisfy $y(2) = y'(2) = 0$.

The analysis so far is straight-forward and it seems likely that it has been carried thus far before. The major difficulty is in the evaluation of the constants A and B. In view of the complexity of the functions involved it is, indeed, remarkable that these constants can be evaluated in a tractable form. The evaluation of the constants is given in the next section.

2. Evaluation of the constants. If $f_1(x), f_2(x)$ denote two functions of x we introduce the usual Wronskian notation $W(f_1, f_2)$ by

$$W(f_1, f_2) = f_1 f_2' - f_2 f_1'. \tag{2.1}$$

In order to satisfy the boundary conditions $y(2) = y'(2) = 0$ we have

$$\begin{aligned} A\, I_1(2) + B\, K_1(2) + P(2) &= 0 \\ A\, I_1'(2) + B\, K_1'(2) + P'(2) &= 0. \end{aligned} \tag{2.2}$$

Since it is well known that $W(I_1(2), K_1(2)) = -\frac{1}{2}$ we have

$$A = 2\, W(P(2), K_1(2)), \quad B = 2\, W(I_1(2), P(2)). \tag{2.3}$$

We evaluate these Wronskians, by the usual procedure, from the differential equations satisfied by $P(x)$ and $I_1(x)$. These differential equations are

$$x\, P'' + P' - (x + x^{-1})\, P = \tfrac{1}{2}x^2(1 - \tfrac{1}{4}x^2) \exp(\tfrac{1}{4}x^2 - 1), \tag{2.4}$$

$$x\, I_1'' + I_1' - (x + x^{-1})\, I_1 = 0. \tag{2.5}$$

We multiply (2.4) by I_1 and (2.5) by P. By subtraction of the resulting equations and integration from $x = 0$ to $x = 2$ we obtain

$$2\, W(I_1(2), P(2)) = \tfrac{1}{2}e^{-1}\int_0^2 x^2(1 - \tfrac{1}{4}x^2)\, e^{x^2/4}\, I_1(x)\, dx. \tag{2.6}$$

Hence

$$B = \tfrac{1}{2}e^{-1}\int_0^2 x^2(1 - \tfrac{1}{4}x^2)\, e^{x^2/4}\, I_1(x)\, dx, \tag{2.7}$$

and similarly

$$A = -\tfrac{1}{2}\, e^{-1}\int_0^2 x^2(1 - \tfrac{1}{4}x^2)e^{x^2/4}\, K_1(x)\, dx. \tag{2.8}$$

In order to evaluate (2.7) we write (2.5) in the form

$$I_1'' + (x^{-1} I_1)' - I_1 = 0. \tag{2.9}$$

Multiplying (2.9) by $\exp(x^2/4)$ and integrating from 0 to 2 we can show, by integrating by parts, that

$$\int_0^2 e^{x^2/4} \left(\tfrac{1}{4}x^2 - 1\right) I_1(x)\,dx = 1 - e\,I_1'(2) + \tfrac{1}{2}\,e\,I_1(2). \tag{2.10}$$

Similarly by multiplying the differential equation by $x^2 \exp(x^2/4)$ and repeating the process we find

$$\int_0^2 e^{x^2/4}\left(x^2 + \tfrac{1}{4}x^4\right) I_1(x)dx = 6\,e\,I_1(2) - 4\,e\,I_1'(2). \tag{2.11}$$

Multiplying (2.10) by eight and subtracting (2.11) we obtain

$$\int_0^2 e^{x^2/4}\left(x^2 - \tfrac{1}{4}x^4\right) I_1(x)dx = 8 - 4e\,I_1'(2) - 2e\,I_1(2) + 8\int_0^2 e^{x^2/4} I_1(x)dx \tag{2.12}$$

From the known recurrence relations of the modified Bessel functions we have

$$2\,I_1'(2) + I_1(2) = 2\,I_0(2). \tag{2.13}$$

Hence

$$\int_0^2 e^{x^2/4}\left(x^2 - \tfrac{1}{4}x^4\right) I_1(x)\,dx = 8 - 4\,e\,I_0(2) + 8\int_0^2 e^{x^2/4} I_1(x)\,dx. \tag{2.14}$$

Let us now consider the integral

$$J = \int_0^2 e^{x^2/4}\, I_1(x)\,dx.$$

The substitution $x = 2u^{\frac{1}{2}}$ transforms J into

$$\begin{aligned} J &= \int_0^1 e^u\, I_1(2\,u^{\frac{1}{2}})\, u^{-\frac{1}{2}}du \\ &= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}\int_0^1 e^u u^n du \\ &= \sum_{n=0}^{\infty} \frac{(1 - n + n(n-1) \ldots (-1)^n n!)e + (-1)^{n+1} n!}{n!(n+1)!} \\ &= e[I_1(2) - I_2(2) + I_3(2) \ldots] + e^{-1} - 1 \\ &= e \sum_{n=1}^{\infty} (-1)^{n+1} I_n(2) + e^{-1} - 1. \end{aligned} \tag{2.15}$$

However, from the generating function for $I_n(x)$ we can prove that

$$e^{-2} = I_0(2) + 2\sum_{n=1}^{\infty} (-1)^n\, I_n(2). \tag{2.16}$$

Thus

$$J = \tfrac{1}{2}e^{-1} + \tfrac{1}{2}eI_0(2) - 1 \tag{2.17}$$

and, from (2.14),

$$\int_0^2 e^{x^2/4}\,(x^2 - \tfrac{1}{4}x^4)\, I_1(x)\, dx = 4e^{-1}. \tag{2.18}$$

Finally from (2.7), (2.18) we have that the constant B is given by

$$B = 2e^{-2}. \tag{2.19}$$

The evaluation of the constant A can also be carried out with the help of the integral representation.

$$2\,K_1\,(2\,u^{\frac{1}{2}})\,u^{-\frac{1}{2}} = \int_0^\infty \exp(-\,uz - z^{-1})dz. \tag{2.20}$$

The final result is that

$$A = e^{-1} + 2e^{-1}\int_0^\infty e^{-z}/(z-1)dz. \tag{2.21}$$

These results imply that the desired solution of (1.7) is

$$y = 2e^{-2}K_1(x) - \tfrac{1}{2}e^{-1}xe^{\frac{1}{4}x^2} - 2e^{-1}\int_0^\infty \frac{z^{\frac{1}{2}}e^{-z}I_1(x(z)^{\frac{1}{2}})dz}{1-z} \tag{2.22}$$

and that the generating function $F(t)$, for q_n is given by

$$F(t) = 2e^{-2}(1-t)^{-\frac{1}{2}}K_1(2(1-t)^{\frac{1}{2}}) - e^{-t} - 2e^{-1}\int_0^\infty H(z,t)dz \tag{2.23}$$

where

$$H(z,t) = z^{\frac{1}{2}}\,e^{-z}\,I_1(2(z-zt)^{\frac{1}{2}})/(1-z)(1-t)^{\frac{1}{2}}.$$

The modified Bessel functions satisfy the well known differentiation formulae

$$\left(\frac{d}{zdz}\right)^m z^{-\alpha}I_\alpha(z) = z^{-\alpha-m}I_{\alpha+m}(z), \tag{2.24}$$

$$\left(\frac{d}{zdz}\right)^m z^{-\alpha}K_\alpha(z) = (-\,1)^m z^{-\alpha-m}K_{\alpha+m}(z). \tag{2.25}$$

Hence

$$q_n = F^{(n)}(0) = 2e^{-2}K_{n+1}(2) + (-\,1)^{n+1} + 2(-\,1)^{n+1}e^{-1}\int_0^\infty M_{n+1}(z)dz, \tag{2.26}$$

where

$$M_{n+1}(z) = z^{\frac{1}{2}(n+1)}\,e^{-z}I_{n+1}\,(2z^{\frac{1}{2}})/(1-z).$$

Since the ménage numbers U_n are given by $U_n = q_n - q_{n-2}$ we find that

$$U_n = 2e^{-2}nK_n(2) + 2(-\,1)^n + 2n(-\,1)^n e^{-1}\int_0^\infty M_n(z)dz. \tag{2.27}$$

If we replace $K_n(2)$, $I_n(2\,z^{\frac{1}{2}})$ by their known series expansions we can obtain an explicit series expression for U_n in terms of n. This expression is very complicated. However (2.27) is a useful expression in that one can derive many of

the known results directly without resorting to the series expression. For example, it is readily shown from (2.27) that

$$\sum_{n=2}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t) - I_0(2t) + I_1(2t). \tag{2.28}$$

Hence, by redefining U_0, U_1, to be 1 and -1 respectively we obtain Touchard's result **(24)**:

$$\sum_{n=0}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t). \tag{2.29}$$

In the next section we shall use (2.27) to derive some new results for the ménage numbers.

3. New results. It has been shown **(11)** that an asymptotic expansion for U_n is given by

$$U_n \sim e^{-2} n!\left[1 - \frac{1}{(n-1)} + \frac{1}{2!(n-1)(n-2)} \cdots\right]. \tag{3.1}$$

By means of (2.27) we shall prove a much deeper result.

To prove this result we write (2.27) in the form

$$U_n = 2e^{-2} n\, K_n(2) + J_n, \tag{3.2}$$

where

$$J_n = 2(-1)^n\left\{1 + n\, e^{-1}\int_0^{\infty} \frac{z^{n/2} e^{-z} I_n(2z^{\frac{1}{2}})}{1-z}\, dz\right\}. \tag{3.3}$$

In (3.3) we replace the first term of the bracket by means of

$$1 = e^{-1}\sum_{m=0}^{\infty} 1/m! \tag{3.4}$$

and $I_n(2\, z^{\frac{1}{2}})$ by its series expression

$$I_n(2z^{\frac{1}{2}}) = z^{\frac{1}{2}n}\sum_{m=0}^{\infty}\frac{z^m}{m!(m+n)!}. \tag{3.5}$$

Hence J_n takes the form

$$J_n = 2(-1)^n e^{-1}\left[\sum_{m=0}^{\infty}\left\{(1/m!) + n\int_0^{\infty}\frac{e^{-z}}{1-z}\sum_{m=0}^{\infty}\frac{z^{m+n}}{m!(m+n)!}\, dz\right\}\right]. \tag{3.6}$$

This can be put in the form

$$J_n = 2(-1)^n e^{-1}\left\{Cn\, I_n(2) + \sum_{m=0}^{\infty}\frac{b_{mn}}{m!(m+n)!}\right\}, \tag{3.7}$$

where

$$\begin{aligned} C &= \int_0^{\infty}\frac{e^{-z}}{1-z}\, dz, \\ b_{mn} &= (m+n)! - n\{(m+n-1)! + (m+n-2)! + \ldots + 1\} \\ &= (m+n-1)!m - n\{(m+n-2)! + (m+n-3)! + \ldots + 1\}. \end{aligned} \tag{3.8}$$

It is trivial to show

$$|C| < 4e^{-1}, \tag{3.9}$$

and

$$|nI_n(2)| < e/(n-1)!. \tag{3.10}$$

Hence

$$|CnI_n(2)| < 4/(n-1)!. \tag{3.11}$$

Let us consider the series term of (3.7) and write

$$\begin{aligned} H_n &= \sum_{m=0}^{\infty} \frac{b_{mn}}{m!(m+n)!} \\ &= \frac{n! - n\{(n-1)! + \ldots + 1)\}}{n!} \\ &\qquad + \frac{(n+1)! - n(n! + (n-1)! + \ldots + 1)}{(n+1)!} \\ &\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!} \\ &= -\frac{(n-2)! + (n-3)! + \ldots + 1}{(n-1)!}\left(1 + \frac{1}{n+1}\right) \\ &\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!}. \end{aligned} \tag{3.12}$$

If $n > 7$ it is easily shown that

$$\frac{(n-2)! + (n-3)! + \ldots + 1}{(n-1)!}\left(1 + \frac{1}{n+1}\right) < \frac{2}{n+1} \tag{3.13}$$

and

$$\left|\sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!}\right| < \frac{2(e-1)}{n+1}. \tag{3.14}$$

Hence for $n > 7$,

$$|H_n| < \frac{2e}{n+1}. \tag{3.15}$$

Actually (3.15) is a very crude inequality. It is, however, sufficient for our purposes.

Combining these results we have from (3.7)

$$|J_n| < \frac{4}{n+1} + \frac{8}{e(n-1)!} \tag{3.16}$$

if $n > 7$.

Hence for $n > 8$ we have

$$|J_n| < 0.45. \tag{3.17}$$

Let us now return to (3.2) and examine the series expression for $K_n(2)$. This is given by

$$K_n(2) = \tfrac{1}{2}\sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} + \tfrac{1}{2}(-1)^n \sum_{m=0}^{\infty} \frac{\Psi(n+m+1)+\Psi(m+1)}{m!(n+m)!}, \tag{3.18}$$

where

$$\Psi(k+1) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{k} - \gamma, \Psi(1) = -\gamma \tag{3.19}$$

and γ is Euler's constant.

It is easily shown that

$$\left| \sum_{m=0}^{\infty} \frac{\Psi(n+m+1)+\Psi(m+1)}{m!(n+m)!} \right| < \frac{e}{2(n-1)!}. \tag{3.20}$$

This implies

$$2n K_n(2) = n \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} + R_n, \tag{3.21}$$

where the remainder satisfies $|R_n| < n\, e/(n-1)!$

Combining the results of (3.2), (3.17) and (3.21) we obtain

$$U_n = e^{-2} n \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} + R_n' \tag{3.22}$$

where for $n > 8$ the remainder R_n' is definitely less than $\frac{1}{2}$.

Using the notation $\{x\}$ to denote the closest integer to x, we have shown that, for $n > 8$

$$U_n = \left\{ e^{-2} n \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} \right\}. \tag{3.23}$$

It is easy to verify that (3.23) remains valid for $0 < n < 7$. Hence we have proved the following theorem:

THEOREM. *For all values of n the ménage numbers U_n are given by* (3.23).

It is thus seen that the asymptotic expansion obtained in **(11)** is much more than an asymptotic expansion.

In concluding this section we might remark that about half of the terms in (3.23) are redundant in that their sum adds up to less than $\frac{1}{2}$. Further our analysis also implies that

$$U_n = \{2e^{-2} n K_n(2)\}. \tag{3.24}$$

We shall make use of (3.24) in the next section to make an interesting conjecture.

4. A Conjecture. The modified Bessel function $K_n(2)$ has the integral representation

$$K_n(2) = \tfrac{1}{2} \int_0^{\infty} t^{n-1} e^{-t-t^{-1}} dt. \tag{4.1}$$

Hence (3.24) may be written

$$U_n = \left\{ e^{-2}\, n \int_0^\infty t^{n-1}\, e^{-t-t^{-1}} dt \right\}. \tag{4.2}$$

The discovery of (4.2) led us to re-examine some of the known results in Latin rectangles. The simplest problem in this class is the so-called "problème des rencontres." This asks for the number of ways R_n of writing a second line of integers $1, 2, \ldots n$ which is discordant with a first line of integers written in their normal order. It is well known that

$$R_n = \{e^{-1}\, n!\} = \left\{ e^{-1} \int_0^\infty x^n\, e^{-x}\, dx \right\}. \tag{4.3}$$

Next in simplicity, in this class of problems, is the so-called reduced three line Latin rectangle problem. This asks for the number of ways P_n of having two lines of integers each of which is discordant with the first line of integers, written in normal order. For this case it was shown by Yamamoto **(26)** that

$$P_n \sim e^{-3}\, (n!)^2 \left[1 + \frac{H_1(-\frac{1}{2})}{n} + \frac{H_2(-\frac{1}{2})}{n(n-1)} + \ldots \right], \tag{4.4}$$

where $H_n(x)$ is a Hermite polynomial.

We have been able to prove an equivalent formula, namely

$$P_n \sim e^{-3}\, (n!) \int_0^\infty x^n\, e^{-x-x^{-1}-x^{-2}}\, dx. \tag{4.5}$$

Finally Erdös and Kaplansky **(7)** have shown that the number P_n^k of reduced (n by $(k+1)$), Latin rectangles is given asymptotically by

$$P_n^k \sim e^{-\frac{1}{2}k(k-1)} (n!)^{k-1} \left[1 - \binom{k}{3} n^{-1} + \left(\frac{1}{2}\binom{k}{3}^2 + \frac{1}{2}\binom{k}{3}(k-5) \right) n^{-2} + \ldots \right] \tag{4.6}$$

for $K < (\log n)^{3/2-\epsilon}$. The validity of the same formula was proved by Yamamoto **(26)** for $k < n^{1/3-\delta}$. The structure of the formula suggests an integral representation of the type

$$P_n^k \sim e^{-\frac{1}{2}k(k-1)} (n!)^{k-2} \int_0^\infty x^n \exp\left(- x - \binom{k}{3} x^{-1} + \frac{1}{2}\binom{k}{3}(k-5)\, x^{-2} + \ldots \right) dx. \tag{4.7}$$

Formula (4.7) is, as we have seen, true for $k = 2{,}3$. If it were possible to prove an integral relation of this type then the asymptotic behavior of P_n^k could be determined for all values of k.

5. An exact expression for the ménage numbers. The usual explicit expression given for the ménage numbers U_n is

$$U_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!. \tag{5.1}$$

In this section we shall derive a second expression from Touchard's generating function (2.9)

$$\sum_{n=0}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t). \tag{5.2}$$

Touchard has remarked that (5.2) constitutes a Neumann expansion for the, function $e^{-2t}/(1-t)$ in terms of the modified Bessel functions $I_n(2t)$. However as far as we are aware, (5.2) has never been inverted to give an explicit expression for the U_n.

If we expand $e^{-2t}/(1-t)$ into a Maclaurin expansion of the form

$$\frac{e^{-2t}}{1-t} = \sum_{r=0}^{\infty} \frac{k_r t^r}{r!}. \tag{5.3}$$

then

$$k_r = \left[\frac{d^r}{dt^r}\frac{e^{-2t}}{1-t}\right]_{t=0} = r! \sum_{s=0}^{r} \frac{(-2)^s}{s!}. \tag{5.4}$$

Further from the well formulae for the coefficients of a Neumann expansion, (5.2) gives

$$U_n = \frac{2(i^n)}{\pi} \int_C \frac{e^{-2t}\, O_n(2it)}{1-t}\, dt, \tag{5.5}$$

where C is any closed contour, enclosing $t = 0$, such that $|t| < 1$. $O_n(z)$ are the so-called Neumann polynomials given explicitly by

$$O_n(z) = \tfrac{1}{4} \sum_{m=0}^{[\frac{1}{2}n]} \frac{n(n-m-1)!\,(\tfrac{1}{2}z)^{2m-n-1}}{m!}. \tag{5.6}$$

It follows immediately from (5.4), (5.5) and (5.6) that

$$U_n = \sum_{m=0}^{[\frac{1}{2}n]} \frac{(-1)^m n(n-m-1)!\, k_{n-2m}}{m!(n-2m)!} \tag{5.7}$$

If we use the umbral convention of replacing k_r by k^r we obtain the neat, mnemonic, formula

$$U_n = 2\, T_n(\tfrac{1}{2}k). \tag{5.8}$$

where $T_n(k)$ is the Chebyshev polynomial.

Table of Ménage Numbers, U_n

n										
0										1
1										−1
2										0
3										1
4										2
5										13
6										80
7										579
8										4738
9										43387
10									4	39792
11									48	90741
12									592	16642
13									7755	96313
14								1	09274	34464
15								16	48064	35783
16								264	93914	69058
17								4522	64356	01207
18								81705	64062	24416
19							15	57461	89109	94665
20							312	40021	86712	53762
21							6577	61864	45769	02053
22						1	45051	25042	12302	24304
23						33	43382	81820	37841	46955
24						803	99425	36462	30706	80706
25						20136	19745	87449	39236	99123
26					5	24412	12770	21518	36760	81296
27					141	80874	54121	35441	26917	90045
28					3976	29238	67612	00144	54828	24194
29				1	15464	79231	29989	49665	85597	50193
30				34	68204	08266	14983	47273	40955	31712
31				1076	37754	44394	44821	25463	33529	40175
32				34481	07559	89439	56929	18585	03293	19426
33			11	39021	31602	21345	03795	43638	02432	51567
34			387	63360	88757	64510	83282	09689	42454	55168
35			13579	25683	97610	83548	12838	24806	55155	91633
36		4	89263	68181	72674	64273	50357	97412	89388	39554
37		181	17111	44161	23578	95013	36816	90501	14249	74653
38		6889	66679	77874	33823	33907	79975	80757	02511	45232
39	2	68887	96926	13377	25044	79310	17322	96268	42696	37331
40	107	62771	05129	32852	47921	55467	77103	56797	10498	56642

Table of Ménage Numbers, U_n.

n											
41		4415	56290	19891	48194	39830	83196	99970	42707	08660	48747
42	1	85566	65097	95828	03659	83212	57515	14716	68334	59763	96848
43	79	83996	94833	63418	59137	63816	96992	08396	12446	35031	15589
44	3514	90268	88496	81285	48747	33216	99334	22942	19228	03980	73090
45										1	58254
		17445	46717	35843	13657	70852	22706	45836	45728	90212	00713
46										72	83366
		69590	77881	51946	38308	62111	11982	87007	50904	26641	27392
47										3424	83522
		40098	53669	54471	72547	68779	40243	86115	44706	26361	30391
48									1	64468	09110
		06041	87840	60152	09219	74918	57252	67810	16409	45316	09090
49									80	62507	03682
		27218	60142	82965	59317	42716	23933	33754	57054	84141	72839
50									4032	96672	76936
		58890	36142	10938	08808	59685	47971	75030	78168	58457	34752
51								2	05765	21900	19435
		33778	81355	42153	61997	39439	63306	20885	84756	93390	60409
52								107	03985	67349	78651
		61744	28069	71363	85025	44933	81813	48096	08655	89107	05410
53								5675	25075	13866	33831
		27158	19299	47659	39404	93066	88177	80601	40960	60903	31861
54							3	06574	69734	91799	35488
		42199	94397	97715	89238	83812	11946	29345	13169	39005	76112
55							168	67497	76536	19957	88857
		92576	17576	87982	88650	64735	97608	01398	80030	29273	13563
56							9448	97804	17604	12841	09142
		94695	41458	72821	08832	08427	22881	97653	69427	61227	49570
57						5	38766	55699	35481	92625	84146
		19035	75909	00166	11666	74714	78157	33164	78496	20304	86819
58						312	58246	74716	90470	28455	64948
		52151	42090	06485	17961	70756	96110	13984	56829	80382	26128
59						18447	94228	72968	63947	37917	08535
		14181	37324	89815	08658	95856	70696	19067	11401	86377	75277
60					11	07199	89841	10584	13191	32048	20675
		94487	31311	19751	93291	00461	34299	17114	32556	45129	57442
61					675	58267	09933	77006	40277	16176	92091
		00422	09590	14676	72585	11993	77206	51822	90952	23565	78401
62					41897	56666	72062	88667	24148	39418	91007
		40473	50277	33700	66678	14090	58913	27371	86013	61045	83552
63				26	40244	43295	64975	42616	59667	86983	89232
		61742	01966	97316	33064	88618	71601	99260	68529	43996	56223
64				1690	18892	81029	16685	73828	37219	43143	46766
		12623	19720	95291	01110	79691	33663	80938	68220	78795	03874
65			1	09889	52094	38550	08753	98369	25269	57562	74720
		74686	09288	18130	98379	85654	60435	38029	33627	62308	89183

NOTE: $U_{45} = 15825417445 \ldots$ *etc.*

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